Notes for an Introductory Course On Electrical Machines and Drives

E.G.Strangas

MSU Electrical Machines and Drives Laboratory

Contents

Preface

The purpose of these notes is be used to introduce Electrical Engineering students to Electrical Machines, Power Electronics and Electrical Drives. They are primarily to serve our students at MSU: they come to the course on Energy Conversion and Power Electronics with a solid background in Electric Circuits and Electromagnetics, and many want to acquire a basic working knowledge of the material, but plan a career in a different area (venturing as far as computer or mechanical engineering). Other students are interested in continuing in the study of electrical machines and drives, power electronics or power systems, and plan to take further courses in the field.

Starting from basic concepts, the student is led to understand how force, torque, induced voltages and currents are developed in an electrical machine. Then models of the machines are developed, in terms of both simplified equations and of equivalent circuits, leading to the basic understanding of modern machines and drives. Power electronics are introduced, at the device and systems level, and electrical drives are discussed.

Equations are kept to a minimum, and in the examples only the basic equations are used to solve simple problems.

These notes do not aim to cover completely the subjects of Energy Conversion and Power Electronics, nor to be used as a reference, not even to be useful for an advanced course. They are meant only to be an aid for the instructor who is working with intelligent and interested students, who are taking their first (and perhaps their last) course on the subject. How successful this endeavor has been will be tested in the class and in practice.

In the present form this text is to be used solely for the purposes of teaching the introductory course on Energy Conversion and Power Electronics at MSU.

E.G.STRANGAS

E. Lansing, Michigan and *Pyrgos, Tinos*

A Note on Symbols

Throughout this text an attempt has been made to use symbols in a consistent way. Hence a script letter, say v denotes a scalar time varying quantity, in this case a voltage. Hence one can see

$$v = 5\sin\omega t \;\text{ or }\; v = \hat{v}\sin\omega t$$

The same letter but capitalized denotes the rms value of the variable, assuming it is periodic. Hence:

$$v = \sqrt{2}V sin\omega t$$

The capital letter, but now bold, denotes a phasor:

$$\mathbf{V} = Ve^{j\theta}$$

Finally, the script letter, bold, denotes a space vector, i.e. a time dependent vector resulting from three time dependent scalars:

$$\mathbf{v} = v_1 + v_2 e^{j\gamma} + v_3 e^{j2\gamma}$$

In addition to voltages, currents, and other obvious symbols we have:

B	Magnetic flux Density (T)
H	Magnetic filed intensity (A/m)
Φ	Flux (Wb) (with the problem that a capital letter is used to show a time dependent scalar)
λ, Λ, $\boldsymbol{\lambda}$	flux linkages (of a coil, rms, space vector)
ω_s	synchronous speed (in electrical degrees for machines with more than two-poles)
ω_o	rotor speed (in electrical degrees for machines with more than two-poles)
ω_m	rotor speed (mechanical speed no matter how many poles)
ω_r	angular frequency of the rotor currents and voltages (in electrical degrees)
T	Torque (Nm)
$\Re(\cdot)$, $\Im(\cdot)$	Real and Imaginary part of $\cdot$

1

Three Phase Circuits and Power

Chapter Objectives

In this chapter you will learn the following:

- The concepts of power, (real reactive and apparent) and power factor
- The operation of three-phase systems and the characteristics of balanced loads in Y and in Δ
- How to solve problems for three-phase systems

1.1 ELECTRIC POWER WITH STEADY STATE SINUSOIDAL QUANTITIES

We start from the basic equation for the instantaneous electric power supplied to a load as shown in figure 1.1

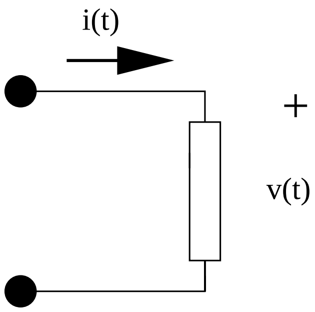

Fig. 1.1 A simple load

$$p(t) = i(t) \cdot v(t) \tag{1.1}$$

where $i(t)$ is the instantaneous value of current through the load and $v(t)$ is the instantaneous value of the voltage across it.

In quasi-steady state conditions, the current and voltage are both sinusoidal, with corresponding amplitudes $\hat{i}$ and $\hat{v}$, and initial phases, ϕ_i and ϕ_v, and the same frequency, $\omega = 2\pi/T - 2\pi f$:

$$v(t) = \hat{v}\sin(\omega t + \phi_v) \tag{1.2}$$

$$i(t) = \hat{i}\sin(\omega t + \phi_i) \tag{1.3}$$

In this case the rms values of the voltage and current are:

$$V = \sqrt{\frac{1}{T}\int_0^T \hat{v}\left[\sin(\omega t + \phi_v)\right]^2 dt} = \frac{\hat{v}}{\sqrt{2}} \tag{1.4}$$

$$I = \sqrt{\frac{1}{T}\int_0^T \hat{i}\left[\sin(\omega t + \phi_i)\right]^2 dt} = \frac{\hat{i}}{\sqrt{2}} \tag{1.5}$$

and these two quantities can be described by phasors, $\mathbf{V} = V^{\angle\phi_v}$ and $\mathbf{I} = I^{\angle\phi_i}$.

Instantaneous power becomes in this case:

$$\begin{aligned} p(t) &= 2VI\left[\sin(\omega t + \phi_v)\sin(\omega t + \phi_i)\right] \\ &= 2VI\frac{1}{2}\left[\cos(\phi_v - \phi_i) + \cos(2\omega t + \phi_v + \phi_i)\right] \end{aligned} \tag{1.6}$$

The first part in the right hand side of equation 1.6 is independent of time, while the second part varies sinusoidally with twice the power frequency. The average power supplied to the load over an integer time of periods is the first part, since the second one averages to zero. We define as real power the first part:

$$P = VI\cos(\phi_v - \phi_i) \tag{1.7}$$

If we spend a moment looking at this, we see that this power is not only proportional to the rms voltage and current, but also to $\cos(\phi_v - \phi_i)$. The cosine of this angle we define as displacement factor, DF. At the same time, and in general terms (i.e. for periodic but not necessarily sinusoidal currents) we define as power factor the ratio:

$$pf = \frac{P}{VI} \tag{1.8}$$

and that becomes in our case (i.e. sinusoidal current and voltage):

$$pf = \cos(\phi_v - \phi_i) \tag{1.9}$$

Note that this is not generally the case for non-sinusoidal quantities. Figures 1.2 - 1.5 show the cases of power at different angles between voltage and current.

We call the power factor leading or lagging, depending on whether the current of the load leads or lags the voltage across it. It is clear then that for an inductive/resistive load the power factor is lagging, while for a capacitive/resistive load the power factor is leading. Also for a purely inductive or capacitive load the power factor is 0, while for a resistive load it is 1.

We define the product of the rms values of voltage and current at a load as apparent power, S:

$$S = VI \tag{1.10}$$

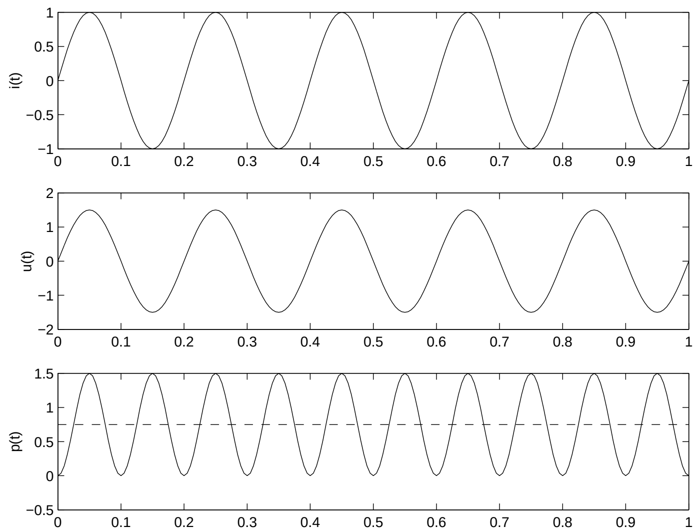

Fig. 1.2 Power at pf angle of 0^o. The dashed line shows average power, in this case maximum

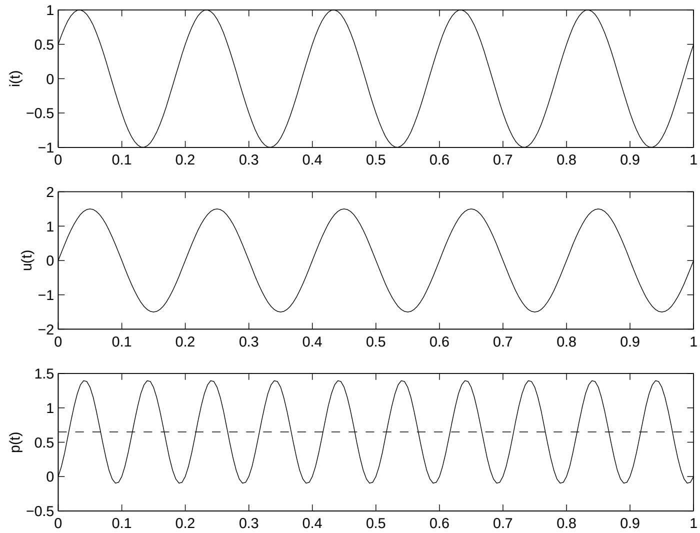

Fig. 1.3 Power at pf angle of 30^o. The dashed line shows average power

and as reactive power, Q

$$Q = VI \sin(\phi_v - \phi_i) \tag{1.11}$$

Reactive power carries more significance than just a mathematical expression. It represents the energy oscillating in and out of an inductor or a capacitor and a source for this energy must exist. Since the energy oscillation in an inductor is 180^0 out of phase of the energy oscillating in a capacitor,

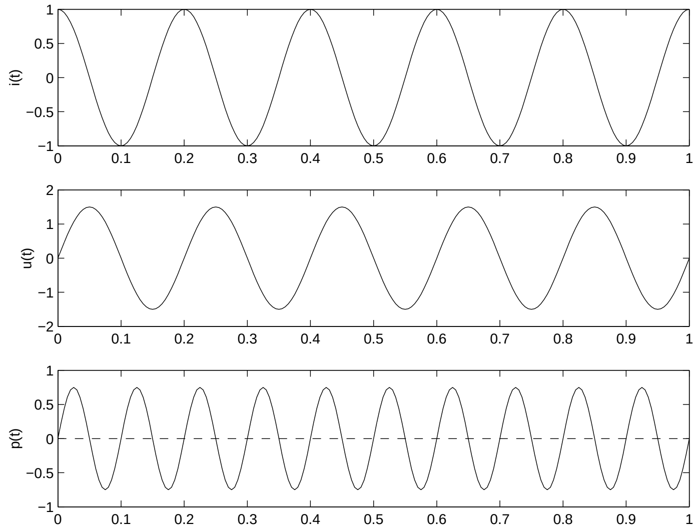

Fig. 1.4 Power at pf angle of 90^o. The dashed line shows average power, in this case zero

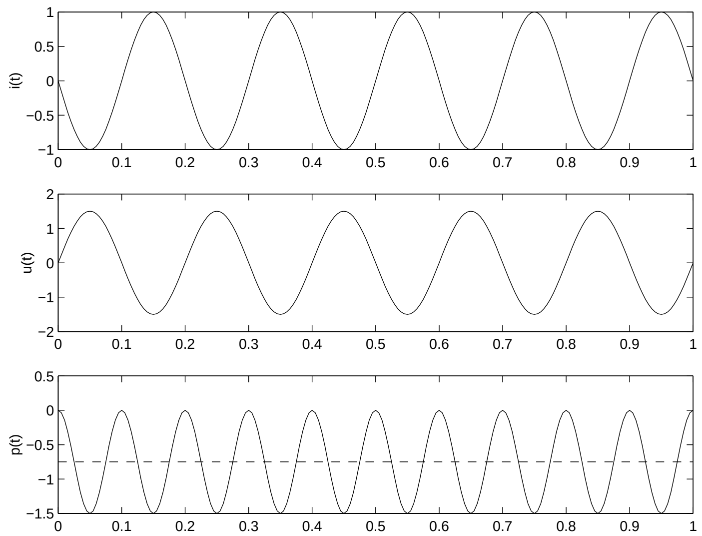

Fig. 1.5 Power at pf angle of 180^o. The dashed line shows average power, in this case negative, the opposite of that in figure 1.2

the reactive power of the two have opposite signs by convention positive for an inductor, negative for a capacitor.

The units for real power are, of course, W, for the apparent power VA and for the reactive power VAr.

Using phasors for the current and voltage allows us to define complex power $\mathbf{S}$ as:

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* \tag{1.12}$$
$$= V^{\angle\phi_v} I^{\angle-\phi_i} \tag{1.13}$$

and finally

$$\mathbf{S} = P + jQ \tag{1.14}$$

For example, when

$$v(t) = \sqrt(2 \cdot 120 \cdot \sin(377t + \frac{\pi}{6})V \tag{1.15}$$
$$i(t) = \sqrt(2 \cdot 5 \cdot \sin(377t + \frac{\pi}{4})A \tag{1.16}$$

then $S = VI = 120 \cdot 5 = 600W$, while $pf = \cos(\pi/6 - \pi/4) = 0.966$ leading. Also:

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = 120^{\angle\pi/6}\, 5^{\angle-\pi/4} = 579.6W - j155.3VAr \tag{1.17}$$

Figure 1.6 shows the phasors for lagging and leading power factors and the corresponding complex power $\mathbf{S}$.

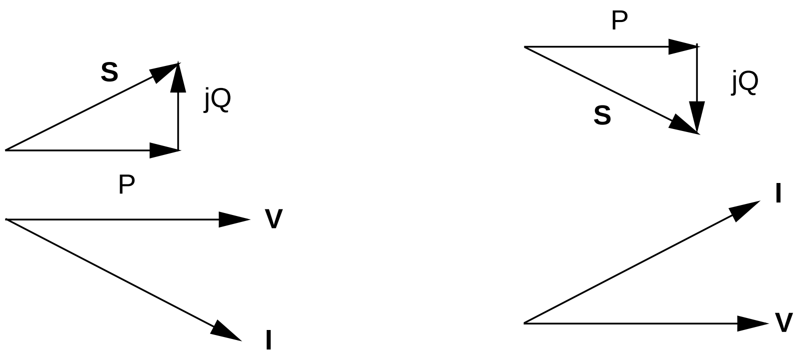

Fig. 1.6 (a) lagging and (b) leading power factor

1.2 SOLVING 1-PHASE PROBLEMS

Based on the discussion earlier we can construct the table below:

Type of load	Reactive power	Power factor
Reactive	$Q > 0$	lagging
Capacitive	$Q < 0$	leading
Resistive	$Q = 0$	1

We also notice that if for a load we know any two of the four quantities, S, P, Q, pf, we can calculate the other two, e.g. if $S = 100kVA$, $pf = 0.8\ leading$, then:

$$
\begin{aligned}
P &= S \cdot pf = 80kW \\
Q &= -S\sqrt{1 - pf^2} = -60kVAr\ ,\ or \\
\sin(\phi_v - \phi_i) &= \sin\left[\arccos 0.8\right] \\
Q &= S\sin(\phi_v - \phi_i)
\end{aligned}
$$

Notice that here $Q < 0$, since the pf is leading, i.e. the load is capacitive.

Generally in a system with more than one loads (or sources) real and reactive power balance, but *not* apparent power, i.e. $P_{total} = \sum_i P_i$, $Q_{total} = \sum_i Q_i$, but $S_{total} \neq \sum_i S_i$.

In the same case, if the load voltage were $V_L = 2000V$, the load current would be $I_L = S/V = 100 \cdot 10^3/2 \cdot 10^3 = 50A$. If we use this voltage as reference, then:

$$
\begin{aligned}
\mathbf{V} &= 2000^{\angle 0}V \\
\mathbf{I} &= 50^{\angle \phi_i} = 50^{\angle 36.9^o}A \\
\mathbf{S} &= \mathbf{V}\,\mathbf{I}^* = 2000^{\angle 0} \cdot 50^{\angle -36.9^o} = P + jQ = 80 \cdot 10^3 W - j60 \cdot 10^3 VAr
\end{aligned}
$$

1.3 THREE-PHASE BALANCED SYSTEMS

Compared to single phase systems, three-phase systems offer definite advantages: for the same power and voltage there is less copper in the windings, and the total power absorbed remains constant rather than oscillate around its average value.

Let us take now three sinusoidal-current sources that have the same amplitude and frequency, but their phase angles differ by 120^0. They are:

$$
\begin{aligned}
i_1(t) &= \sqrt{2}I\sin(\omega t + \phi) \\
i_2(t) &= \sqrt{2}I\sin(\omega t + \phi - \frac{2\pi}{3}) \\
i_3(t) &= \sqrt{2}I\sin(\omega t + \phi + \frac{2\pi}{3})
\end{aligned}
\tag{1.18}
$$

If these three current sources are connected as shown in figure 1.7, the current returning though node n is zero, since:

$$
\sin(\omega t + \phi) + \sin(\omega t - \phi + \frac{2\pi}{3}) + \sin(\omega t + \phi + \frac{2\pi}{3}) \equiv 0 \tag{1.19}
$$

Let us also take three voltage sources:

$$
\begin{aligned}
v_a(t) &= \sqrt{2}V\sin(\omega t + \phi) \\
v_b(t) &= \sqrt{2}V\sin(\omega t + \phi - \frac{2\pi}{3}) \\
v_c(t) &= \sqrt{2}V\sin(\omega t + \phi + \frac{2\pi}{3})
\end{aligned}
\tag{1.20}
$$

connected as shown in figure 1.8. If the three impedances at the load are equal, then it is easy to prove that the current in the branch $n - n'$ is zero as well. Here we have a first reason why

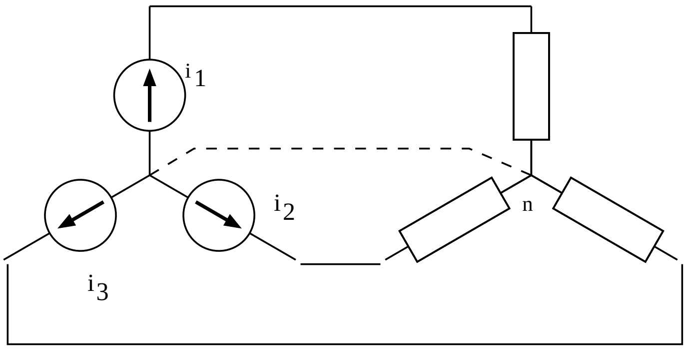

Fig. 1.7 Zero neutral current in a Y-connected balanced system

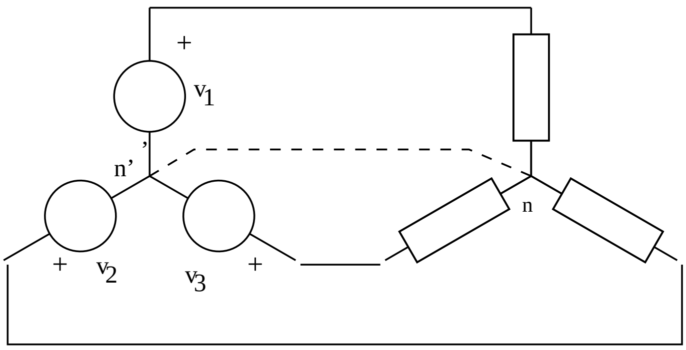

Fig. 1.8 Zero neutral current in a voltage-fed, Y-connected, balanced system.

three-phase systems are convenient to use. The three sources together supply three times the power that one source supplies, but they use three wires, while the one source alone uses two. The wires of the three-phase system and the one-phase source carry the same current, hence with a three-phase system the transmitted power can be tripled, while the amount of wires is only increased by 50%.

The loads of the system as shown in figure 1.9 are said to be in Y or star. If the loads are connected as shown in figure 1.11, then they are said to be connected in Delta, Δ, or triangle. For somebody who cannot see beyond the terminals of a Y or a Δ load, but can only measure currents and voltages there, it is impossible to discern the type of connection of the load. We can therefore consider the two systems equivalent, and we can easily transform one to the other without any effect outside the load. Then the impedances of a Y and its equivalent Δ symmetric loads are related by:

$$Z_Y = \frac{1}{3} Z_\Delta \tag{1.21}$$

Let us take now a balanced system connected in Y, as shown in figure 1.9. The voltages between the neutral and each of the three phase terminals are $\mathbf{V_{1n}} = V^{\angle\phi}$, $\mathbf{V_{2n}} = V^{\angle\phi - \frac{2\pi}{3}}$, and $\mathbf{V_{3n}} = V^{\angle\phi + \frac{2\pi}{3}}$. Then the voltage between phases 1 and 2 can be shown either through trigonometry or vector geometry to be:

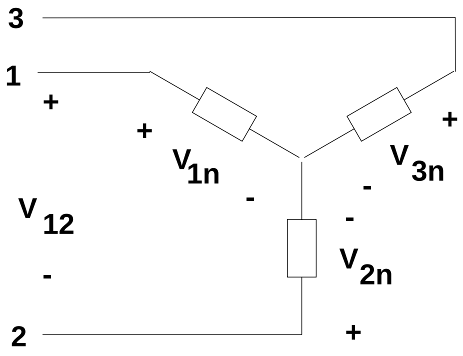

Fig. 1.9 Y Connected Loads: Voltages and Currents

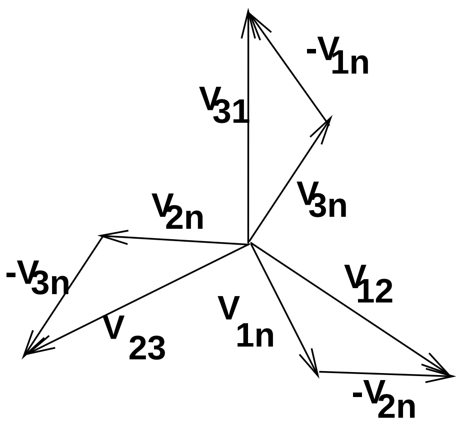

Fig. 1.10 Y Connected Loads: Voltage phasors

$$\mathbf{V_{12}} = \mathbf{V_1} - \mathbf{V_2} = \sqrt{3}V^{\angle\phi + \frac{\pi}{3}} \tag{1.22}$$

This is shown in the phasor diagrams of figure 1.10, and it says that the rms value of the line-to-line voltage at a Y load, V_{ll}, is $\sqrt{3}$ times that of the line-to-neutral or phase voltage, V_{ln}. It is obvious that the phase current is equal to the line current in the Y connection. The power supplied to the system is three times the power supplied to each phase, since the voltage and current amplitudes and the phase differences between them are the same in all three phases. If the power factor in one phase is $pf = \cos(\phi_v - \phi_i)$, then the total power to the system is:

$$\begin{aligned} \mathbf{S_{3\phi}} &= P_{3\phi} + jQ_{3\phi} \\ &= 3\mathbf{V_1}\mathbf{I_1^*} \\ &= \sqrt{3}V_{ll}I_l\cos(\phi_v - \phi_i) + j\sqrt{3}V_{ll}I_l\sin(\phi_v - \phi_i) \end{aligned} \tag{1.23}$$

Similarly, for a connection in Δ, the phase voltage is equal to the line voltage. On the other hand, if the phase currents phasors are $\mathbf{I_{12}} = I^{\angle\phi}$, $\mathbf{I_{23}} = I^{\angle\phi - \frac{2\pi}{3}}$ and $\mathbf{I_{31}} = I^{\angle\phi + \frac{2\pi}{3}}$, then the current of

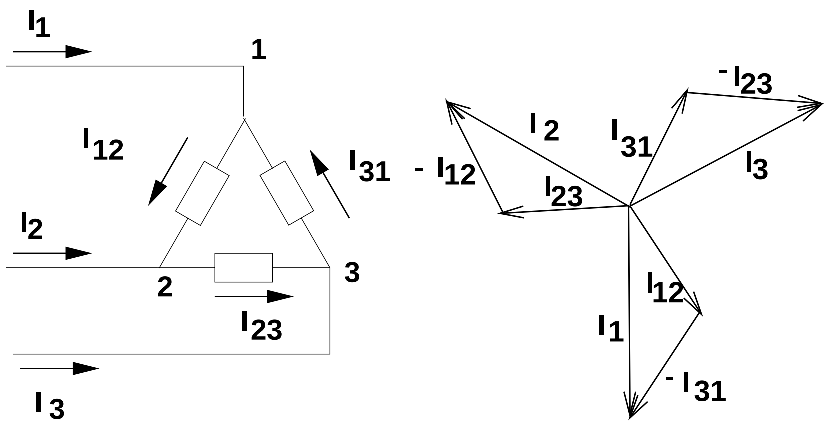

Fig. 1.11 Δ Connected Loads: Voltages and Currents

line 1, as shown in figure 1.11 is:

$$\mathbf{I_1} = \mathbf{I_{12}} - \mathbf{I_{31}} = \sqrt{3} I^{\angle \phi - \frac{\pi}{3}} \tag{1.24}$$

To calculate the power in the three-phase, Y connected load,

$$\begin{aligned} \mathbf{S_{3\phi}} &= P_{3\phi} + jQ_{3\phi} \\ &= 3\mathbf{V_1 I_1^*} \\ &= \sqrt{3} V_{ll} I_l \cos(\phi_v - \phi_i) + j\sqrt{3} V_{ll} I_l \sin(\phi_v - \phi_i) \end{aligned} \tag{1.25}$$

1.4 CALCULATIONS IN THREE-PHASE SYSTEMS

It is often the case that calculations have to be made of quantities like currents, voltages, and power, in a three-phase system. We can simplify these calculations if we follow the procedure below:

1. transform the Δ circuits to Y,
2. connect a neutral conductor,
3. solve one of the three 1-phase systems,
4. convert the results back to the Δ systems.

1.4.1 Example

For the 3-phase system in figure 1.12 calculate the line-line voltage, real power and power factor at the load.

First deal with only one phase as in the figure 1.13:

$$\begin{aligned} \mathbf{I} &= \frac{120}{j1 + 7 + j5} = 13.02^{\angle -40.6^o} A \\ \mathbf{V_{ln}} &= \mathbf{I\ Z_l} = 13.02^{\angle -40.6^o}(7 + j5) = 111.97^{\angle -5^o} V \\ \mathbf{S_{L,1\phi}} &= \mathbf{V_L\ I^*} = 1.186 \cdot 10^3 + j0.847 \cdot 10^3 \\ P_{L1\phi} &= 1.186 kW, \quad Q_{L1\phi} = 0.847 kVAr \\ pf &= \cos(-5^o - (-40.6^o)) = 0.814\ lagging \end{aligned}$$

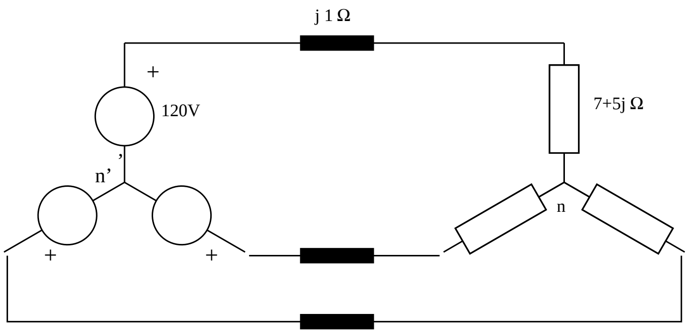

Fig. 1.12 A problem with Y connected load.

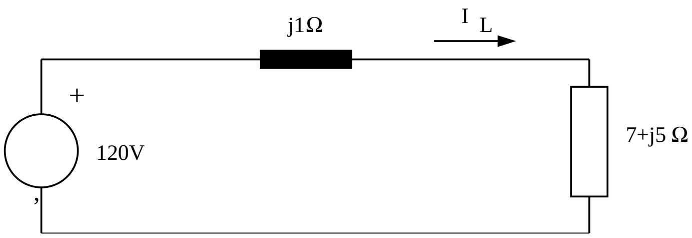

Fig. 1.13 One phase of the same load

For the three-phase system the load voltage (line-to-line), and real and reactive power are:

$$
\begin{aligned}
V_{L,l-l} &= \sqrt{3} \cdot 111.97 = 193.94V \\
P_{L,3\phi} &= 3.56kW, \qquad Q_{L,3\phi} = 2.541kVAr
\end{aligned}
$$

1.4.2 Example

For the system in figure 1.14, calculate the power factor and real power at the load, as well as the phase voltage and current. The source voltage is $400V$ *line-line.*

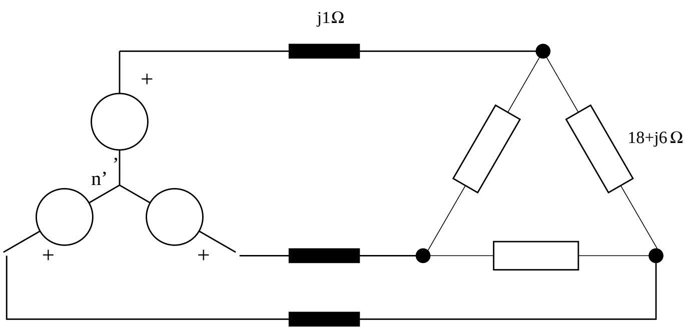

Fig. 1.14 Δ-connected load

First we convert the load to Y and work with one phase. The line to neutral voltage of the source is $V_{ln} = 400/\sqrt{3} = 231V$.

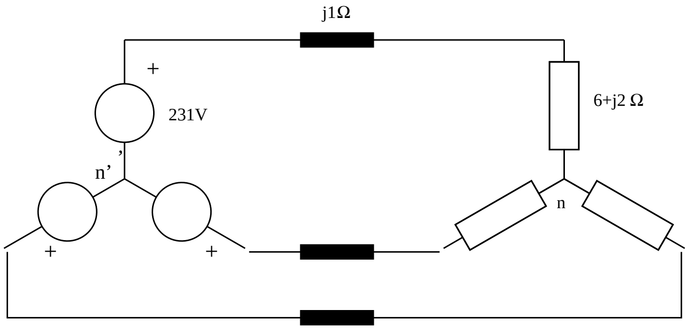

Fig. 1.15 The same load converted to Y

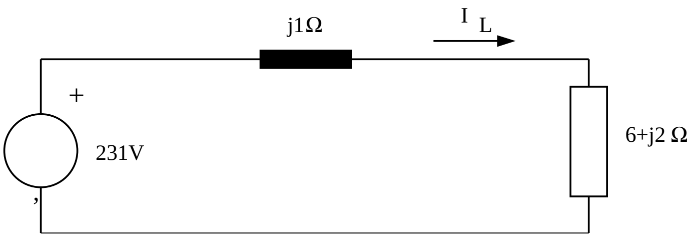

Fig. 1.16 One phase of the Y load

$$\begin{aligned} \mathbf{I_L} &= \frac{231}{j1+6+j2} = 34.44^{\angle -26.6^o} A \\ \mathbf{V_L} &= \mathbf{I_L}(6+j2) = 217.8^{\angle -8.1^o} V \end{aligned}$$

The power factor at the load is:

$$pf = \cos(\phi_v - \phi_i) = \cos(-8.1^o + 26.6^o) = 0.948 lag$$

Converting back to Δ:

$$\begin{aligned} I_\phi &= I_L/\sqrt{3} = 34.44/\sqrt{3} = 19.88A \\ V_{ll} &= 217.8 \cdot \sqrt{3} \cdot 377.22V \end{aligned}$$

At the load

$$P_{3\phi} = \sqrt{3} V_{ll}\, I_L\, pf = \sqrt{3} \cdot 377.22 \cdot 34.44 \cdot 0.948 = 21.34kW$$

1.4.3 Example

Two loads are connected as shown in figure 1.17. Load 1 draws from the system $P_{L1} = 500kW$ at 0.8 pf lagging, while the total load is $S_T = 1000kVA$ at 0.95 pf lagging. What is the pf of load 2?

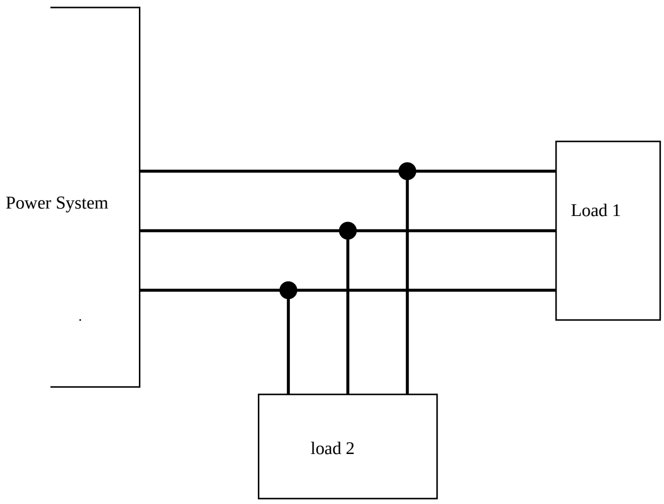

Fig. 1.17 Two loads fed from the same source

Note first that for the total load we can add real and reactive power for each of the two loads:

$$\begin{aligned} P_T &= P_{L1} + P_{L2} \\ Q_T &= Q_{L1} + Q_{L2} \\ S_T &\neq S_{L1} + S_{L2} \end{aligned}$$

From the information we have for the total load

$$\begin{aligned} P_T &= S_T pf_T = 950kW \\ Q_T &= S_T \sin(\cos^{-1} 0.95) = 312.25kVAr \end{aligned}$$

Note positive Q_T since pf is lagging

For the load $L1$, $P_{L1} = 500kW$, $pf_1 = 0.8$ lag,

$$\begin{aligned} S_{L1} &= \frac{500 \cdot 10^3}{0.8} = 625kVA \\ Q_{L1} &= \sqrt{S_{L1}^2 - P_{L1}^2} = 375kVAr \end{aligned}$$

Q_{L1} is again positive, since pf is lagging.

Hence,

$$P_{L2} = P_T - P_{L1} = 450kW \tag{1.26}$$
$$Q_{L2} = Q_T - Q_{L1} = -62.75kVAr$$

and

$$pf_{L2} = \frac{P_{L2}}{S_{L2}} = \frac{450}{\sqrt{420^2 + 62.75^2}} = 0.989\, leading.$$

Notes

- A sinusoidal signal can be described uniquely by:

  1. as e.g. $v(t) = 5\sin(2\pi ft + \phi_v)$,
  2. by its graph,
  3. as a phasor and the associated frequency.

  one of these descriptions is enough to produce the other two. As an exercise, convert between phasor, trigonometric expression and time plots of a sinusoid waveform.

- It is the phase difference that is important in power calculations, not phase. The phase alone of a sinusoidal quantity does not really matter. We need it to solve circuit problems, after we take one quantity (a voltage or a current) as reference, i.e. we assign to it an arbitrary value, often 0. There is no point in giving the phase of currents and voltages as answers, and, especially for line-line voltages or currents in Δ circuits, these numbers are often wrong and anyway meaningless.

- In both 3-phase and 1-phase systems the sum of the real power and the sum of the reactive power of individual loads are equal respectively to the real and reactive power of the total load. This is not the case for apparent power and of course not for power factor.

- Of the four quantities, real power, reactive power, apparent power and power factor, any two describe a load adequately. The other two can be calculated from them.

- To calculate real reactive and apparent Power when using formulae 1.7, 1.10 1.11 we have to use absolute not complex values of the currents and voltages. To calculate complex power using 1.12 we do use complex currents and voltages and find directly both real and reactive power.

- When solving a circuit to calculate currents and voltages, use complex impedances, currents and voltages.

- Notice two different and equally correct formulae for 3-phase power.

2

Magnetics

Chapter Objectives

In this chapter you will learn the following:

- How Maxwell's equations can be simplified to solve simple practical magnetic problems
- The concepts of saturation and hysteresis of magnetic materials
- The characteristics of permanent magnets and how they can be used to solve simple problems
- How Faraday's law can be used in simple windings and magnetic circuits
- Power loss mechanisms in magnetic materials
- How force and torque is developed in magnetic fields

2.1 INTRODUCTION

Since a good part of electromechanical energy conversion uses magnetic fields it is important early on to learn (or review) how to solve for the magnetic field quantities in simple geometries and under certain assumptions. One such assumption is that the frequency of all the variables is low enough to neglect all displacement currents. Another is that the media (usually air, aluminum, copper, steel etc.) are homogeneous and isotropic. We'll list a few more assumptions as we move along.

2.2 THE GOVERNING EQUATIONS

We start with Maxwell's equations, describing the characteristics of the magnetic field at low frequencies. First we use:

$$\nabla \cdot \mathbf{B} = 0 \tag{2.1}$$

the integral form of which is:

$$\int \mathbf{B} \cdot d\mathbf{A} \equiv 0 \tag{2.2}$$

for any path. This means that there is no source of flux, and that all of its lines are closed.

Secondly we use

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_A \mathbf{J} \cdot d\mathbf{A} \tag{2.3}$$

where the closed loop is defining the boundary of the surface A. Finally, we use the relationship between $\mathbf{H}$, the strength of the magnetic field, and $\mathbf{B}$, the induction or flux density.

$$\mathbf{B} = \mu_r \mu_0 \mathbf{H} \tag{2.4}$$

where μ_0 is the permeability of free space, $4\pi 10^{-7} Tm/A$, and μ_r is the relative permeability of the material, 1 for air or vacuum, and a few hundred thousand for magnetic steel.

There is a variety of ways to solve a magnetic circuit problem. The equations given above, along with the conditions on the boundary of our geometry define a *boundary value problem*. Analytical methods are available for relatively simple geometries, and numerical methods, like Finite Elements Analysis, for more complex geometries.

Here we'll limit ourselves to very simple geometries. We'll use the equations above, but we'll add boundary conditions and some more simplifications. These stem from the assumption of existence of an *average flux path* defined within the geometry. Let's tackle a problem to illustrate it all. In

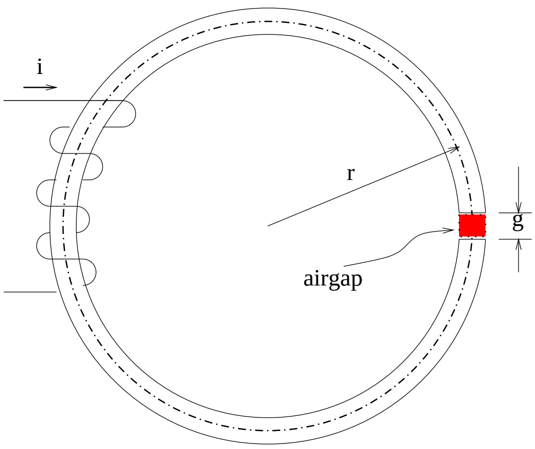

Fig. 2.1 A simple magnetic circuit

figure 2.1 we see an iron ring with cross section A_c, average diameter r, that has a gap of length g and a coil around it of N turns, carrying a current i. The additional assumptions we'll make in order to calculate the magnetic field density everywhere are:

- The magnetic flux remains within the iron and a tube of air, the airgap, defined by the cross section of the iron and the length of the gap. This tube is shown in dashed lines in the figure.
- The flux flows parallel to a line, the average flux path, shown in dash-dot.

- Flux density is uniform at any cross-section and perpendicular to it.

Following one flux line, the one coinciding with the average path, we write:

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int \mathbf{J} \cdot d\mathbf{A} \tag{2.5}$$

where the second integral extends over any surface (a bubble) terminating on the path of integration. But equation 2.2, together with the first assumption assures us that for any cross section of the geometry the flux, $\Phi = \int_{A_c} \mathbf{B} \cdot d\mathbf{A} = B_{avg} A_c$, is constant. Since both the cross section and the flux are the same in the iron and the air gap, then

$$\begin{aligned} B_{iron} &= B_{air} \\ \mu_{iron} H_{iron} &= \mu_{air} H_{air} \end{aligned} \tag{2.6}$$

and finally

$$\begin{aligned} H_{iron}(2\pi r - g) + H_{gap} \cdot g = Ni \\ \left[\frac{\mu_{air}}{\mu_{iron}}(2\pi r - g) + g\right] H_{gap} = Ni \end{aligned}$$

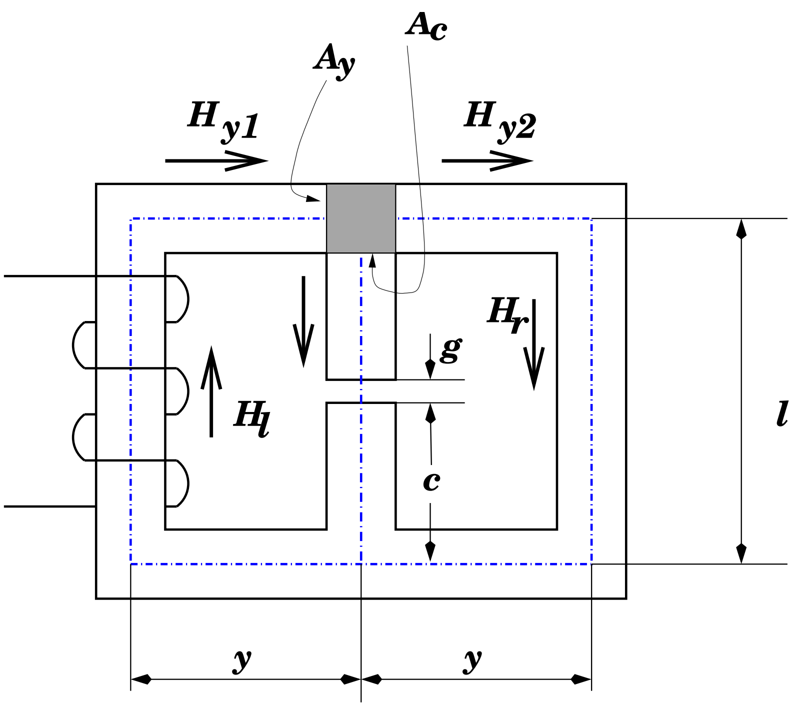

Fig. 2.2 A slightly complex magnetic circuit

Let us address one more problem: calculate the magnetic field in the airgap of figure 2.2, representing an iron core of depth d . Here we have to use two loops like the one above, and we have a choice of possible three. Taking the one that includes the legs of the left and in the center, and the outer one, we can write:

$$\begin{aligned} H_l \cdot l + H_{y1} \cdot y + H_c \cdot c + H_g \cdot g + H_c \cdot c + H_{y1} \cdot y &= Ni \\ H_l \cdot h + 2H_{y1} \cdot y + H_r \cdot l + 2H_{y2} \cdot y &= Ni \end{aligned} \tag{2.7}$$

Applying equation 2.3 to the closed surface shown shaded we also obtain:

$$B_l\, A_y - B_c\, A_c - B_r A_y = 0$$

and of course

$$B_l = \mu\, H_l, \qquad B_c = \mu\, H_c, \qquad B_r = \mu\, H_r \qquad B_g = \mu_0 H_g$$

The student can complete the problem. We notice though something interesting: a similarity between Kirchoff's equations and the equations above. If we decide to use:

$$\Phi = B\,A \tag{2.8}$$

$$\mathcal{R} = \frac{l}{A\mu} \tag{2.9}$$

$$\mathcal{F} = Ni \tag{2.10}$$

then we notice that we can replace the circuits above with the one in figure 2.3, with the following correspondence:

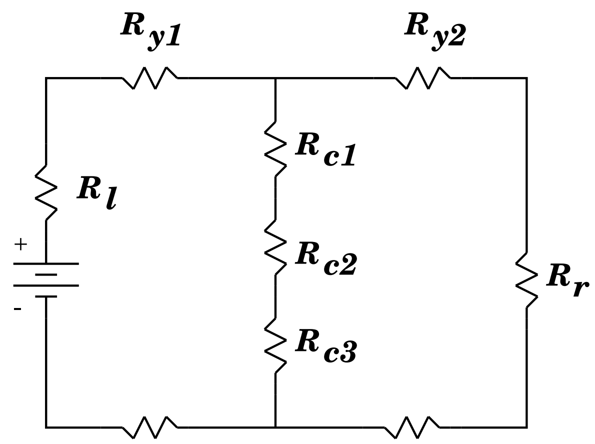

Fig. 2.3 Equivalent electric circuit for the magnetic circuit in figure 2.2

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Magnetic	Electrical
$\mathcal{F}$, magnetomotive force	V, voltage, or electromotive force
Φ, flux	I, current
$\mathcal{R}$, reluctance	R, resistance

This is of course a great simplification for students who have spent a lot of effort on electrical circuits, but there are some differences. One is the nonlinearity of the media in which the magnetic field lives, particularly ferrous materials. This nonlinearity makes the solution of direct problems a little more complex (problems of the type: for given flux find the necessary current) and the inverse problems more complex and sometimes impossible to solve without iterations (problems of the type: for given currents find the flux).

2.3 SATURATION AND HYSTERESIS

Although for free space a equation 2.3 is linear, in most ferrous materials this relationship is nonlinear. Neglecting for the moment hysteresis, the relationship between H and B can be described by a curve of the form shown in figure 2.4. From this curve, for a given value of B or H we can find the other one and calculate the permeability $\mu = B/H$.

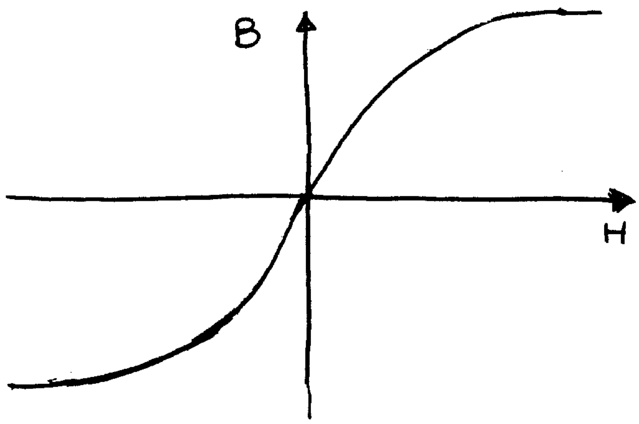


Fig. 2.4 Saturation in ferrous materials

In addition to the phenomenon of saturation we have also to study the phenomenon of hysteresis in ferrous materials. The defining difference is that if saturation existed alone, the flux would be a unique function of the field intensity. When hysteresis is present, flux density for a give value of field intensity, H depends also on the history of magnetic flux density, B in it. We can describe the relationship between field intensity, H and flux density B in homogeneous, isotropic steel with the curves of 2.5. These curves show that the flux density depends on the history of the magnetization of the material. This dependence on history is called hysteresis. If we replace the curve with that of the locus of the extrema, we obtain the saturation curve of the iron, which in itself can be quite useful.

Going back to one of the curves in 2.5, we see that when the current changes sinusoidally between the two values, $\hat{\imath}$ and $-\hat{\imath}$, then the point corresponding to (H, B) travels around the curve. During this time, power is transferred to the iron, referred to as hysteresis losses, P_{hyst}. The energy of these losses for one cycle is proportional to the area inside the curve. Hence the power of the losses is proportional to this surface, the frequency, and the volume of iron; it increases with the maximum value of B:

$$P_{hyst} = kf\hat{B}^x \quad 1 < x < 2 \tag{2.11}$$

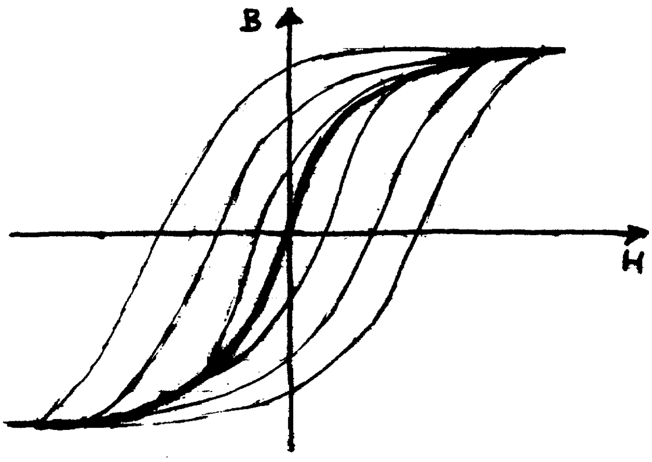


Fig. 2.5 Hysteresis loops and saturation

If the value of H, when increasing towards $\hat{H}$, does so not monotonously, but at one point, H_1, decreases to H_2 and then increases again to its maximum value, $\hat{H}$, a minor hysteresis loop is created, as shown in figure 2.6. The energy lost in one cycle includes these additional minor loop surfaces.

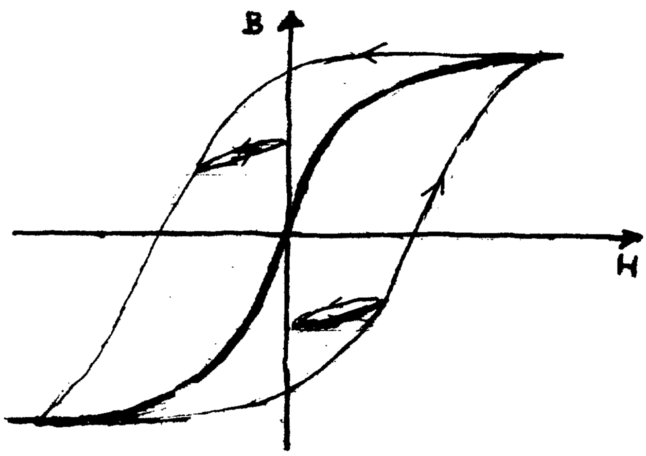


Fig. 2.6 Minor loops on a hysteresis curve

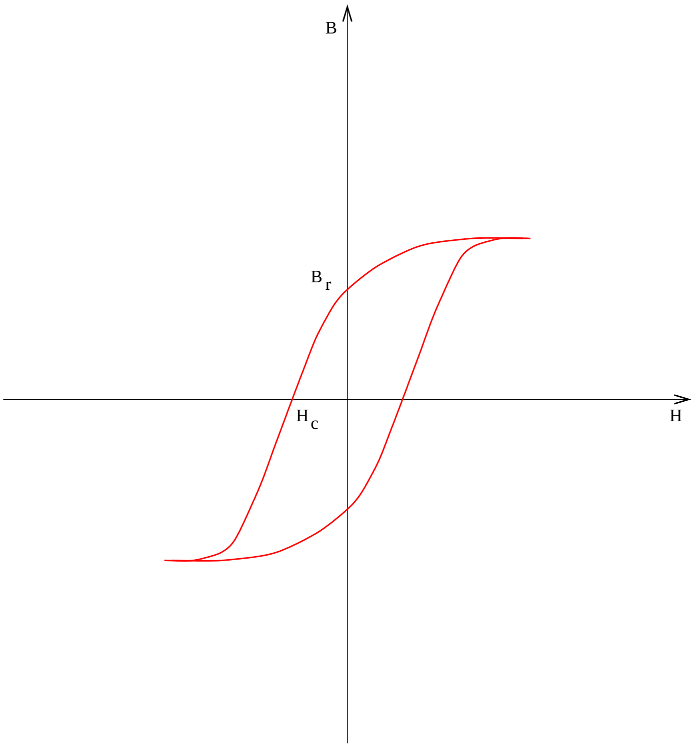


Fig. 2.7 Hysteresis curve in magnetic steel

2.4 PERMANENT MAGNETS

If we take a ring of iron with uniform cross section and a magnetic characteristic of the material that in figure 2.7, and one winding around it, and look only at the second quadrant of the curve, we notice that for $H = 0$, i.e. no current in an winding there will be some nonzero flux density, B_r. In addition, it will take current in the winding pushing flux in the opposite direction (negative current) in order to make the flux zero. The iron in the ring has became a permanent magnet. The value of the field intensity at this point is $-H_c$. In practice a permanent magnet is operating not at the second quadrant of the hysteresis loop, but rather on a minor loop, as shown on figure 2.6 that can be approximated with a straight line. Figure 2.8 shows the characteristics of a variety of permanent magnets. The curve of a permanent magnet can be described by a straight line in the region of interest, 2.9, corresponding to the equation:

$$B_m = \frac{H_m + H_c}{H_c} B_r \tag{2.12}$$

2.4.1 Example

In the magnetic circuit of figure 2.10 the length of the magnet is $l_m = 1cm$, the length of the air gap is $g = 1mm$ and the length of the iron is $l_i = 20cm$. For the magnet $B_r = 1.1T$, $H_c = 750kA/m$. What is the flux density in the air gap if the iron has infinite permeability and the cross section is uniform?

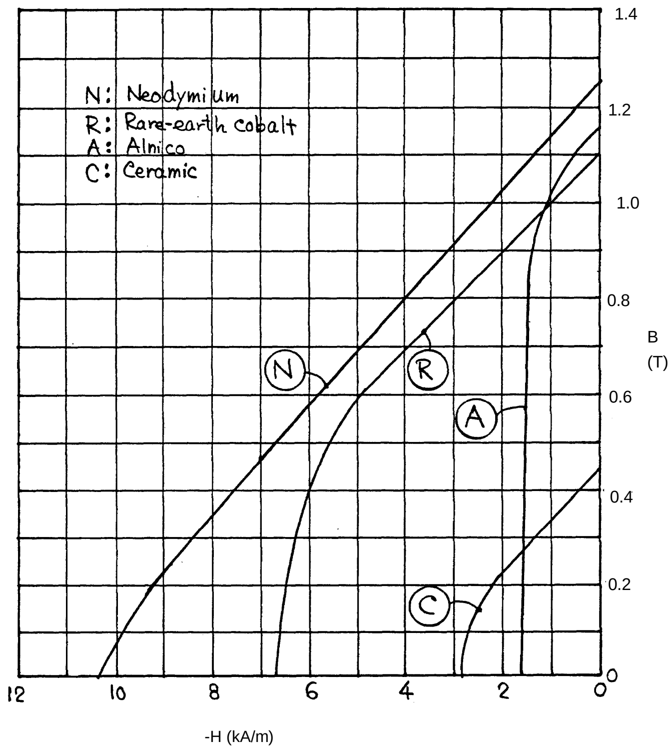


Fig. 2.8 Minor loops on a hysteresis curve

Since the cross section is uniform, B is the same everywhere, and there is no current:

$$H_i \cdot 0.2 + H_g \cdot g + H_m \cdot l_i = 0$$

for infinite iron permeability $H_i = 0$, hence,

$$\begin{aligned} B_{air} \frac{1}{\mu_o} g + (B_m - 1.1) \left(\frac{H_c}{B_r} \right) l_i = 0 \\ \Rightarrow B \cdot 795.77 + (B - 1.1) \cdot 6818 = 0 \\ B = 0.985T \end{aligned}$$

2.5 FARADAY'S LAW

We'll see now how voltage is generated in a coil and the effects it may have on a magnetic material. This theory, along with the previous chapter, is essential in calculating the transfer of energy through a magnetic field.

First let's start with the governing equation again. When flux through a coil changes for whatever reason (e.g. change of the field or relative movement), a voltage is induced in this coil. Figure 2.11

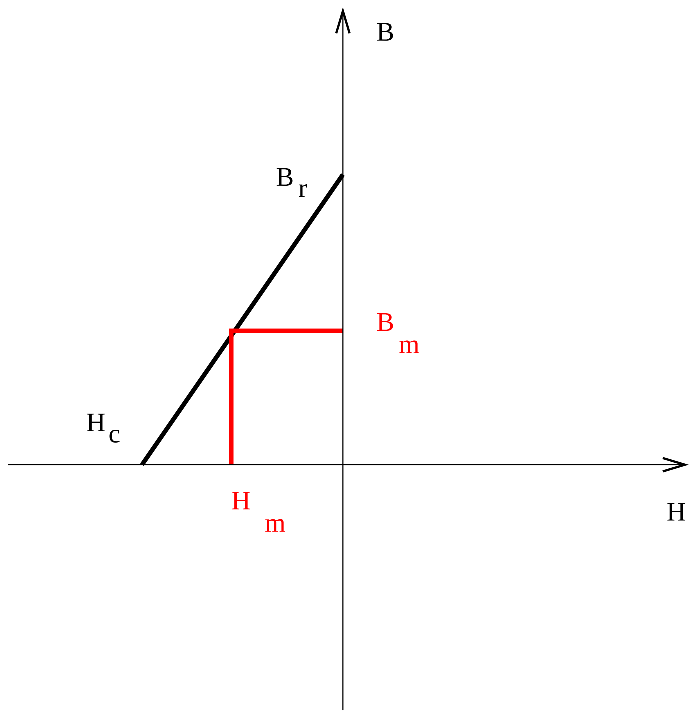


Fig. 2.9 Finding the flux density in a permanent magnet

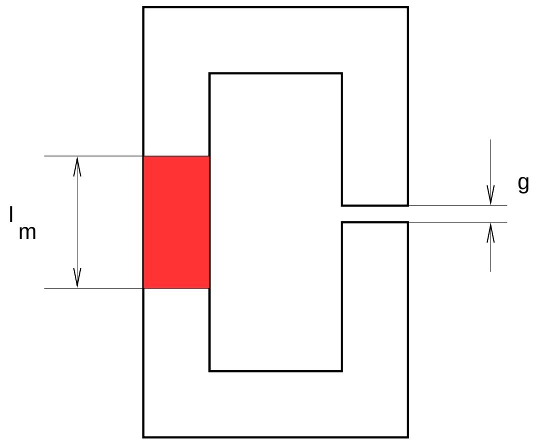


Fig. 2.10 Magnetic circuit for Example 2.4.1

shows such a typical case. Faraday's law relates the electric and magnetic fields. In its integral form:

$$\oint_C \mathcal{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_A \mathbf{B} \cdot d\mathbf{A} \tag{2.13}$$

and in the cases we study it becomes:

$$v(t) = \frac{d\Phi(t)}{dt} \tag{2.14}$$

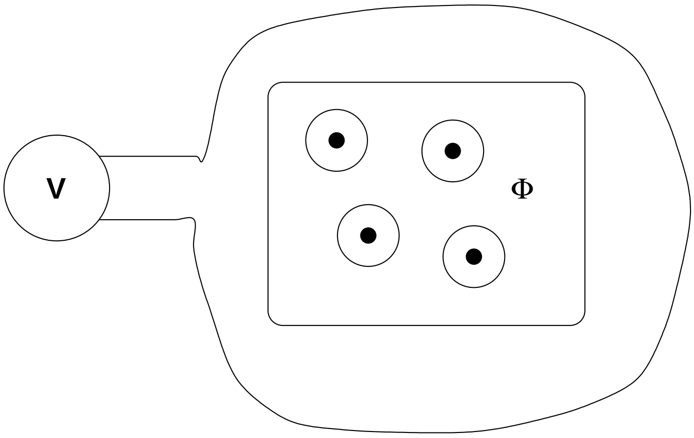


Fig. 2.11 Flux through a coil

If a coil has more than one turns in series, we define as flux linkages of the coil, λ, the sum of the flux through each turn,

$$\lambda = \sum_i \Phi_i \tag{2.15}$$

and then:

$$v(t) = \frac{d\lambda(t)}{dt} \tag{2.16}$$

2.5.1 Example

For the magnetic circuit shown below $\mu_{iron} = \mu_o \cdot 10^5$, the air gap is $1mm$ and the length of the iron core at the average path is $1m$. The cross section of the iron core is $0.04m^2$. The winding labelled 'primary' has 500 turns. A sinusoidal voltage of $60Hz$ is applied to it. What should be the rms value of it if the flux density in the iron (rms) is $0.8T$? What is the current in the coil? The voltage induced in the coil will be

$$\begin{aligned}
\textit{But if} \qquad B(t) = \hat{B}\sin(2\pi f t) &\Rightarrow \Phi(t) = A\hat{B}\sin(2\pi f t) \\
&\Rightarrow \Phi(t) = 0.04(\sqrt{2}\cdot 0.8)\sin(377t)Wb \\
e_1(t) &= \frac{d\Phi}{dt} \\
\Rightarrow e_1(t) = 500\left[0.04\sqrt{2}\cdot 0.8\cdot 377\sin(377t+\frac{\pi}{2})\right] & \quad V \\
\Rightarrow E_1 = \frac{\hat{e_1}}{\sqrt{2}} &= 500\cdot 0.04\cdot 0.8\cdot 377 = 6032V
\end{aligned}$$

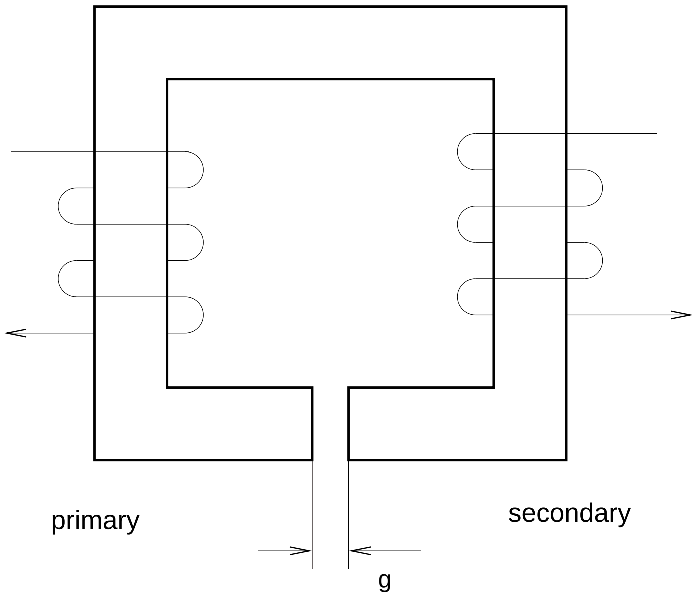


Fig. 2.12 Magnetic circuit for Example 2.5.1

To calculate the current we integrate around the loop of the average path:

$$H_{iron}l + H_{air}g = Ni$$

$$b_{iron} = B_{air} = \sqrt{2}\cdot 0.8\sin(377t) \quad \Rightarrow \quad H_{air} = \frac{\sqrt{2}\cdot 0.8}{\mu_o}\sin(377t)A/m$$

$$\Rightarrow \quad H_{iron} = \frac{\sqrt{2}\cdot 0.8}{\mu_o \cdot 10^5}\sin(377t)A/m$$

Finally

$$500\cdot i = \frac{\sqrt{2}\cdot 0.8\sin(377t)}{\mu_o}\left(\frac{1}{10^5} + \frac{1\cdot 10^{-3}}{1}\right)$$

$$\Rightarrow i = 1.819\sin(377t)A \Rightarrow I = \frac{\hat{i}}{\sqrt{2}} = 1.286A$$

2.6 EDDY CURRENTS AND EDDY CURRENT LOSSES

When the flux through a solid ferrous material varies with time, currents are induced in it. Figure 2.13 gives a simple explanation: Let's consider a ring of iron *defined within the material* shown in black and a flux Φ through it, shown in grey. As the flux changes, a voltage $e = d\Phi/dt$ is induced in the ring. Since the ring is shorted, and has a resistance R, a current flows in it, and Joule losses, $P_{eddy} = e^2/R$, result. We can consider a multitude of such rings in the material, resulting into Joule losses, but the method discussed above is not the appropriate one to calculate these losses. We can, though, estimate that for sinusoidal flux, the flux, voltage, and losses are:

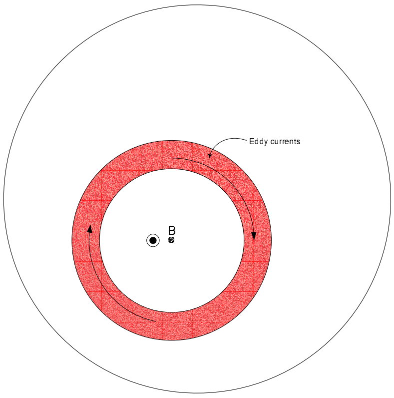


Fig. 2.13 Eddy currents in solid iron

$$\Phi = \hat{\Phi}\sin(\omega\, t) = A\hat{B}\sin(\omega\, t) \tag{2.17}$$

$$e = \omega\hat{\Phi}\cos(\omega\, t) = 2\pi A f \hat{B}\cos(\omega\, t) \tag{2.18}$$

$$P_{eddy} = k\, f^2 \hat{B}^2 \tag{2.19}$$

which tells us that the losses are proportional to the square of both the flux density and frequency. A typical way to decrease losses is to laminate the material, as shown in figure 2.14, decreasing the paths of the currents and the total flux through them.

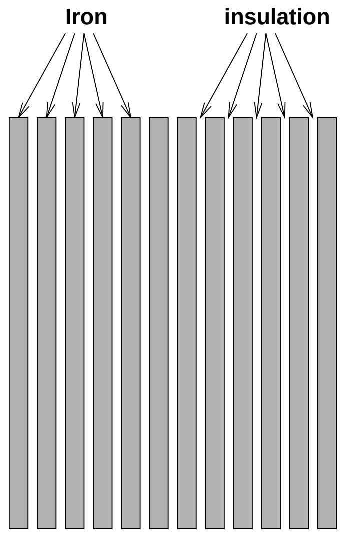


Fig. 2.14 Laminated steel

2.7 TORQUE AND FORCE

Calculating these is quite more complex, since Maxwell's equations do not refer directly to them. The most reasonable approach is to start from energy balance. Then the energy in the firles W_f is the sum of the energy that entered through electrical and mechanical sources.

$$W_f = \sum W_e + \sum W_m \tag{2.20}$$

This in turn can lead to the calculation of the forces since

$$\sum_{k=1}^{K} f_k dx_k = \sum_{j=1}^{J} e_j i_j dt - dW_f \tag{2.21}$$

Hence for a small movement, dx_k, the energies in the equation should be evaluated and from these, forces (or torques), f_k, calculated.

Alternatively, although starting from the same principles, one can use the Maxwell stress tensor to find forces or torques on enclosed volumes, calculate forces using the Lorenz force equation, here $F = liB$, or use directly the balance of energy. Here we'll use only this last method, e.g. balance the mechanical and electrical energies.

In a mechanical system with a force F acting on a body and moving it at velocity v in its direction, the power P_{mech} is

$$P_{mech} = F \cdot v \tag{2.22}$$

This eq. 2.22, becomes for a rotating system with torque T, rotating a body with angular velocity ω_{mech}:

$$P_{mech} = T \cdot w_{mech} \tag{2.23}$$

On the other hand, an electrical source e, supplying current i to a load provides electrical power P_{elec}

$$P_{elec} = e \cdot i \tag{2.24}$$

Since power has to balance, if there is no change in the field energy,

$$P_{elec} = P_{mech} \ \Rightarrow \ T \cdot w_{mech} = e \cdot i \tag{2.25}$$

Notes

- It is more reasonable to solve magnetic circuits starting from the integral form of Maxwell's equations than finding equivalent resistance, voltage and current. This also makes it easier to use saturation curves and permanent magnets.
- Permanent magnets do not have flux density equal to B_R. Equation 2.12defines the relation between the variables, flux density B_m and field intensity H_m in a permanent magnet.
- There are two types of iron losses: eddy current losses that are proportional to the square of the frequency and the square of the flux density, and hysteresis losses that are proportional to the frequency and to some power x of the flux density.

3

Transformers

Although transformers have no moving parts, they are essential to electromechanical energy conversion. They make it possible to increase or decrease the voltage so that power can be transmitted at a voltage level that results in low costs, and can be distributed and used safely. In addition, they can provide matching of impedances, and regulate the flow of power (real or reactive) in a network.

In this chapter we'll start from basic concepts and build the equations and circuits corresponding first to an ideal transformer and then to typical transformers in use. We'll introduce and work with the per unit system and will cover three-phase transformers as well.

After working on this chapter, you'll be able to:

- Choose the correct rating and characteristics of a transformer for a specific application,
- Calculate the losses, efficiency, and voltage regulation of a transformer under specific operating conditions,
- Experimentally determine the transformer parameters given its ratings.

3.1 DESCRIPTION

When we see a transformer on a utility pole all we see is a cylinder with a few wires sticking out. These wires enter the transformer through bushings that provide isolation between the wires and the tank. Inside the tank there is an iron core linking coils, most probably made with copper, and insulated. The system of insulation is also associated with that of cooling the core/coil assembly. Often the insulation is paper, and the whole assembly may be immersed in insulating oil, used to both increase the dielectric strength of the paper and to transfer heat from the core-coil assembly to the outer walls of the tank to the air. Figure 3.1 shows the cutout of a typical distribution transformer

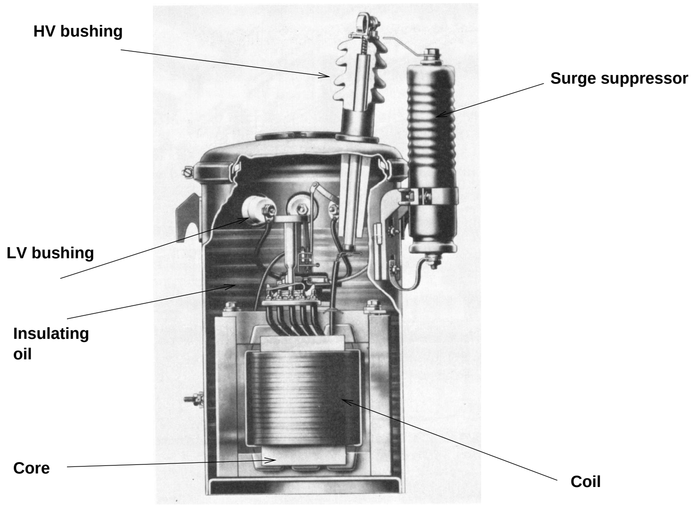


Fig. 3.1 Cutaway view of a single phase distribution transformer. Notice only one HV bushing and lightning arrester

3.2 THE IDEAL TRANSFORMER

Few ideal versions of human constructions exist, and the transformer offers no exception. An ideal transformer is based on very simple concepts, and a large number of assumptions. This is the transformer one learns about in high school.

Let us take an iron core with infinite permeability and two coils wound around it (with zero resistance), one with N_1 and the other with N_2 turns, as shown in figure 3.2. All the magnetic flux is to remain in the iron. We assign *dots* at one terminal of each coil in the following fashion: if the flux

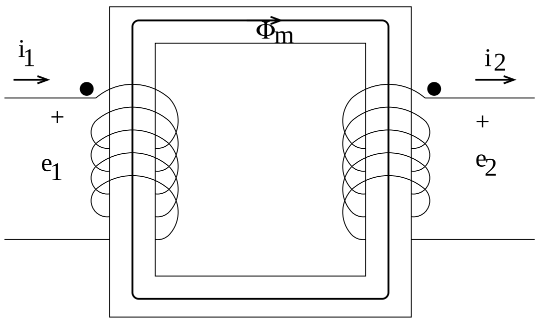


Fig. 3.2 Magnetic Circuit of an ideal transformer

in the core changes, inducing a voltage in the coils, and the dotted terminal of one coil is positive with respect its other terminal, so is the dotted terminal of the other coil. Or, the corollary to this, current into dotted terminals produces flux in the same direction.

Assume that somehow a time varying flux, $\Phi(t)$, is established in the iron. Then the flux linkages in each coil will be $\lambda_1 = N_1\Phi(t)$ and $\lambda_2 = N_2\Phi(t)$. Voltages will be induced in these two coils:

$$e_1(t) = \frac{d\lambda_1}{dt} = N_1\frac{d\Phi}{dt} \tag{3.1}$$

$$e_2(t) = \frac{d\lambda_2}{dt} = N_2\frac{d\Phi}{dt} \tag{3.2}$$

and dividing:

$$\frac{e_1(t)}{e_2(t)} = \frac{N_1}{N_2} \tag{3.3}$$

On the other hand, currents flowing in the coils are related to the field intensity H. If currents flowing in the direction shown, i_1 into the dotted terminal of coil 1, and i_2 out of the dotted terminal of coil 2, then:

$$N_1 \cdot i_1(t) - N_2 i_2(t) = H \cdot l \tag{3.4}$$

but $B = \mu_{iron}H$, and since B is finite and μ_{iron} is infinite, then $H = 0$. We recognize that this is practically impossible, but so is the existence of an ideal transformer.

Finally:

$$\frac{i_1}{i_2} = \frac{N_2}{N_1} \tag{3.5}$$

Equations 3.3 and 3.5 describe this ideal transformer, a two port network. The symbol of a network that is defined by these two equations is in the figure 3.3. An ideal transformer has an

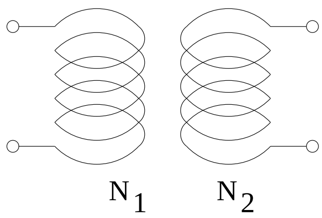


Fig. 3.3 Symbol for an ideal transformer

interesting characteristic. A two-port network that contains it and impedances can be replaced by an equivalent other, as discussed below. Consider the circuit in figure 3.4a. Seen as a two port network

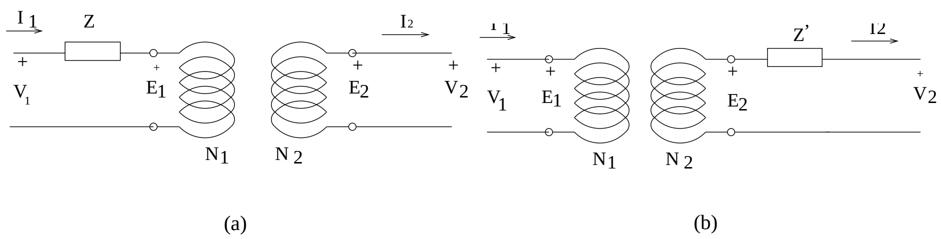


Fig. 3.4 Transferring an impedance from one side to the other of an ideal transformer

with variables v_1, i_1, v_2, i_2, we can write:

$$e_1 = u_1 - i_1 Z \tag{3.6}$$
$$e_2 = \frac{N_2}{N_1} e_1 = \frac{N_2}{N_1} u_1 - \frac{N_2}{N_1} i_1 Z \tag{3.7}$$
$$v_2 = e_2 = \frac{N_2}{N_1} e_1 = \frac{N_2}{N_1} u_1 - i_2 \left(\frac{N_2}{N_1}\right)^2 Z \tag{3.8}$$

which could describe the circuit in figure 3.4b. Generally a circuit on a side 1 can be transferred to side 2 by multiplying its component impedances by $(N_2/N_1)^2$, the voltage sources by (N_2/N_1) and the current sources by (N_1/N_2), while keeping the topology the same.

3.3 EQUIVALENT CIRCUIT

To develop the equivalent circuit for a transformer we'll gradually relax the assumptions that we had first imposed. First we'll relax the assumption that the permeability of the iron is infinite. In that case equation 3.4 does not revert to 3.5, but rather it becomes:

$$N_1 i_1 - N_2 i_2 = \mathcal{R}\Phi_m \tag{3.9}$$

where $\mathcal{R}$ is the reluctance of the path around the core of the transformer and Φ_m the flux on this path. To preserve the ideal transformer equations as part of our new transformer, we can split i_1 to two components: one i_1', will satisfy the ideal transformer equation, and the other, $i_{1,ex}$ will just balance the right hand side. Figure 3.5 shows this.

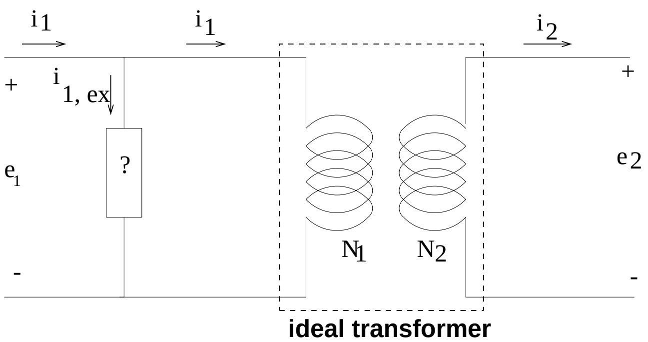


Fig. 3.5 First step to include magnetizing current

$$i_1 = i_1' + i_{1,ex} \tag{3.10}$$
$$N_1 i_{1,ex} = \mathcal{R}\Phi_m \tag{3.11}$$
$$N_1 i_1(t) - N_2 i_2(t) = H \cdot l \tag{3.12}$$

We can replace the current source, $i_{1,ex}$, with something simpler if we remember that the rate of change of flux Φ_m is related to the induced voltage e_1:

$$e_1 = N_1 \frac{d\Phi_m}{dt} \tag{3.13}$$

$$= N_1 \frac{d\left(N_1 i_{1,ex}/\mathcal{R}\right)}{dt} \tag{3.14}$$

$$= \left(\frac{N_1^2}{\mathcal{R}}\right) \frac{di_{1,ex}}{dt} \tag{3.15}$$

Since the current $i_{1,ex}$ flows through something, where the voltage across it is proportional to its derivative, we can consider that this something could be an inductance. This idea gives rise to the equivalent circuit in figure 3.6, where $L_m = \frac{N_1^2}{\mathcal{R}}$ Let us now relax the assumption that all the flux has

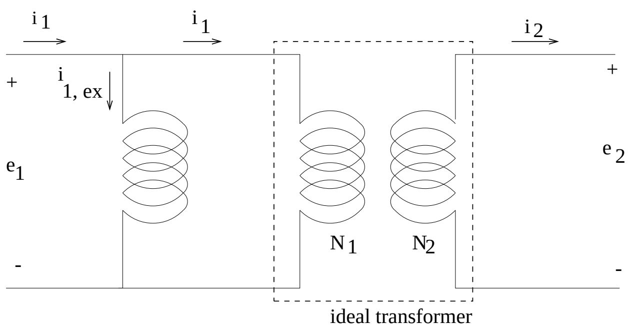


Fig. 3.6 Ideal transformer plus magnetizing branch

to remain in the iron as shown in figure 3.7. Let us call the flux in the iron Φ_m, magnetizing flux, the flux that leaks out of the core and links only coil 1, Φ_{l1}, leakage flux 1, and for coil 2, Φ_{l2}, leakage flux 2. Since Φ_{l1} links only coil 1, then it should be related only to the current there, and the same should be true for the second leakage flux.

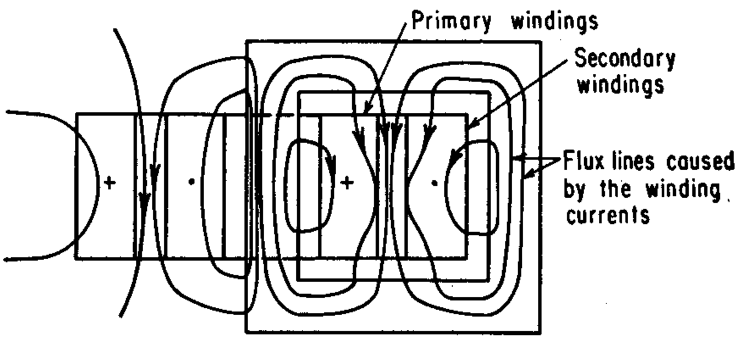


Fig. 3.7 If the currents in the two windings were to have cancelling values of $N \cdot i$, then the only flux left would be the leakage fluxes. This is the case shown here, designed to point out these fluxes.

$$\Phi_{l1} = N_1 i_1 / \mathcal{R}_{l1} \tag{3.16}$$
$$\Phi_{l2} = N_2 i_2 / \mathcal{R}_{l2} \tag{3.17}$$

where $\mathcal{R}_{l1}$ and $\mathcal{R}_{l2}$ correspond to paths that are partially in the iron and partially in the air. As these currents change, so do the leakage fluxes, and a voltage is induced in each coil:

$$e_1 = \frac{d\lambda_1}{dt} = N_1 \left(\frac{d\Phi_m}{dt} \right) + N_1 \frac{d\Phi_{l1}}{dt} = e_1 + \left(\frac{N_1^2}{\mathcal{R}_{l1}} \right) \frac{di_1}{dt} \tag{3.18}$$
$$e_2 = \frac{d\lambda_2}{dt} = N_2 \left(\frac{d\Phi_m}{dt} \right) + N_2 \frac{d\Phi_{l2}}{dt} = e_2 + \left(\frac{N_2^2}{\mathcal{R}_{l2}} \right) \frac{di_2}{dt} \tag{3.19}$$

If we define $L_{l1} \doteq \frac{{N_1}^2}{\mathcal{R}_{l1}}$, $L_{l2} \doteq \frac{N_1^2}{\mathcal{R}_{l2}}$, then we can arrive to the equivalent circuit in figure 3.8. To this

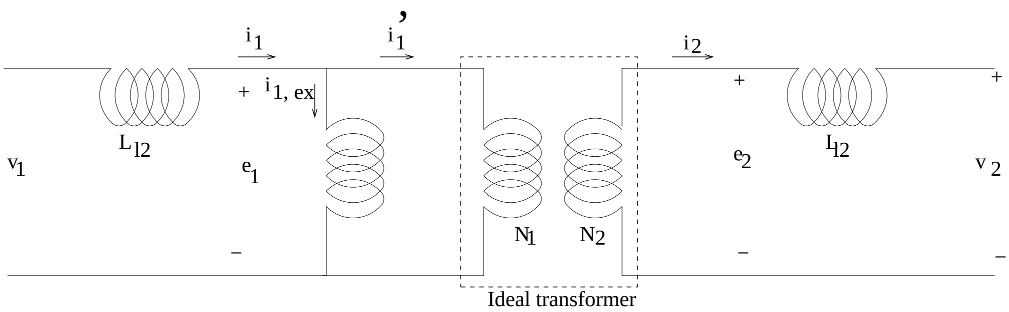


Fig. 3.8 Equivalent circuit of a transformer plus magnetizing and leakage inductances

circuit we have to add:

1. The winding (ohmic) resistance in each coil, $R_{1,wdg}$, $R_{2,wdg}$, with losses $P_{1,wdg} = i_1^2 R_{1,wdg}$, $P_{22,wdg} = i_2^2 R_{2,wdg}$, and

2. some resistance to represent iron losses. These losses (at least the eddy-current ones) are proportional to the square of the flux. But the flux is proportional to the square of the induced voltage e_1, hence $P_{iron} = k e_1^2$. Since this resembles the losses of a resistance supplied by voltage e_1, we can develop the equivalent circuit 3.9.

3.3.1 Example

Let us now use this equivalent circuit to solve a problem. Assume that the transformer has a turns ratio of 4000/120, with $R_{1,wdg} = 1.6\Omega$, $R_{2,wdg} = 1.44m\Omega$, $L_{l1} = 21mH$, $L_{l2} = 19\mu H$, $R_c = 160k\Omega$, $L_m = 450H$. assume that the voltage at the low voltage side is $60Hz$, $V_2 = 120V$, and the power there is $P_2 = 20kW$, at $pf = 0.85$ lagging. Calculate the voltage at the high voltage side and the efficiency of the transformer.

$$X_m = L_m * 2\pi 60 = 169.7k\Omega$$
$$X_1 = 7.92\Omega$$
$$X_2 = 7.16m\Omega$$

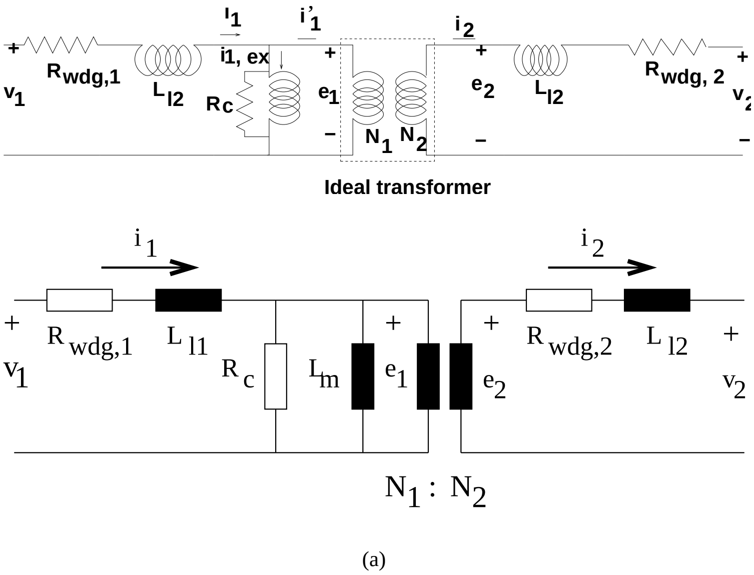


Fig. 3.9 Equivalent circuit for a real transformer

From the power the load:

$$\mathbf{I_2} = P_L/(V_L pf)^{\angle -31.8^0} = 196.1336^{\angle -31.8^0} A$$
$$\mathbf{E}_2 = \mathbf{V}_2 + \mathbf{I}_2\left(R_{wdg,2} + jX_{l2}\right) = 120.98 + j1.045V$$
$$\mathbf{E}_1 = \left(\frac{N_1}{N_2}\right)\mathbf{E}_2 = 4032.7 + j34.83V$$
$$\mathbf{I}_1' = \left(\frac{N_2}{N_1}\right)\mathbf{I}_2 = 5.001 - j3.1017A$$
$$\mathbf{I}_{1,ex} = \mathbf{E}_1\left(\frac{1}{R_c} + \frac{1}{jX_m}\right) = 0.0254 - j0.0236A$$
$$\mathbf{I}_1 = \mathbf{I}_{1,ex} + \mathbf{I}_1' = 5.0255 - j3.125A$$
$$\mathbf{V}_1 = \mathbf{E}_1 + \mathbf{I_1}\left(R_{wdg,1} + jX_{l,1}\right) = 4065.5 + j69.2V = 4066^{\angle 0.9^0} V$$

The power losses are concentrated in the windings and core:

$$P_{wdg,2} = I_2^2 R_{wdg,2} = 196.13^2 \cdot 1.44 \cdot 10^{-3} = 55.39W$$
$$P_{wdg,1} = I_1^2 R_{wdg,1} = 5.918^2 \cdot 1.6 = 56.04W$$
$$P_{core} = E_1^2/R_c = 4032.8^2/(160 \cdot 10^3) = 101.64W$$
$$P_{loss} = P_{wdg,1} + P_{wdg,2} + P_{core} = 213.08W$$
$$\eta = \frac{P_{out}}{P_{in}} = \frac{P_{out}}{(P_{out} + P_{loss})} = \frac{20kW}{20kW + 213.08W} = 0.9895$$

3.4 LOSSES AND RATINGS

Again for a given frequency, the power losses in the core (iron losses) increase with the voltage e_1 (or e_2). These losses cannot be allowed to exceed a limit, beyond which the temperature of the hottest spot in the transformer will rise above the point that will decrease dramatically the life of the insulation. Limits therefore are put to E_1 and E_2 (with a ratio of N_1/N_2), and these limits are the voltage limits of the transformer.

Similarly, winding Joule losses have to be limited, resulting in limits to the currents I_1 and I_2.

Typically a transformer is described by its rated voltages, E_{1N} and E_{2N}, that give both the limits and turns ratio. The ratio of the rated currents, I_{1N}/I_{2N}, is the inverse of the ratio of the voltages if we neglect the magnetizing current. Instead of the transformer rated currents, a transformer is described by its rated apparent power:

$$S_N = E_{1N} I_{1N} = E_{2N} I_{2N} \tag{3.20}$$

Under rated conditions, i.e. maximum current and voltage, in typical transformers the magnetizing current $I_{1,ex}$, does not exceed 1% of the current in the transformer. Its effect therefore on the voltage drop on the leakage inductance and winding resistance is negligible.

Under maximum (rated) current, total voltage drops on the winding resistances and leakage inductances do not exceed in typical transformers 6% of the rated voltage. The effect therefore of the winding current on the voltages E_1 and E_2 is small, and their effect on the magnetizing current can be neglected.

These considerations allow us to modify the equivalent circuit in figure 3.9, to obtain the slightly inaccurate but much more useful equivalent circuits in figures 3.10a, b, and c.

3.4.1 Example

Let us now use these new equivalent circuits to solve the previous problem 3.3.1. We'll use the circuit in 3.10b. Firs let's calculate the combined impedances:

$$R_{wdg} = R_{wdg,1} + \left(\frac{N_1}{N_2}\right)^2 R_{wdg,2} = 3.2\Omega$$
$$X_l = X_{l,1} + \left(\frac{N_1}{N_2}\right)^2 X_{l,2} = 15.8759\Omega$$

then, we solve the circuit:

$$\mathbf{I_2} = P_L/(V_L \cdot pf)^{\angle -31.8^0} = 196.1336^{\angle -31.8^0} A$$
$$\mathbf{E}_2 = \mathbf{V}_2$$
$$\mathbf{I}_1' = \mathbf{I}_2 \cdot \left(\frac{N_2}{N_1}\right) = 5 + j3.102A$$
$$\mathbf{E}_1 = \mathbf{E}_2 \cdot \left(\frac{N_1}{N_2}\right) = 4000V$$
$$\mathbf{I}_{1,ex} = \mathbf{E}_1 \left(\frac{1}{R_c} + \frac{1}{jX_m}\right) = 0.0258 - j0.0235A$$
$$\mathbf{I}_1 = \mathbf{I}_{1,ex} + \mathbf{I}_1' = 5.0259 - j3.125A$$
$$\mathbf{V}_1 = \mathbf{E}_1 + \mathbf{I}_1' \left(R_{wdg} + jX_l\right) = 4065 + j69.45V = 4065^{\angle 1^0} V$$

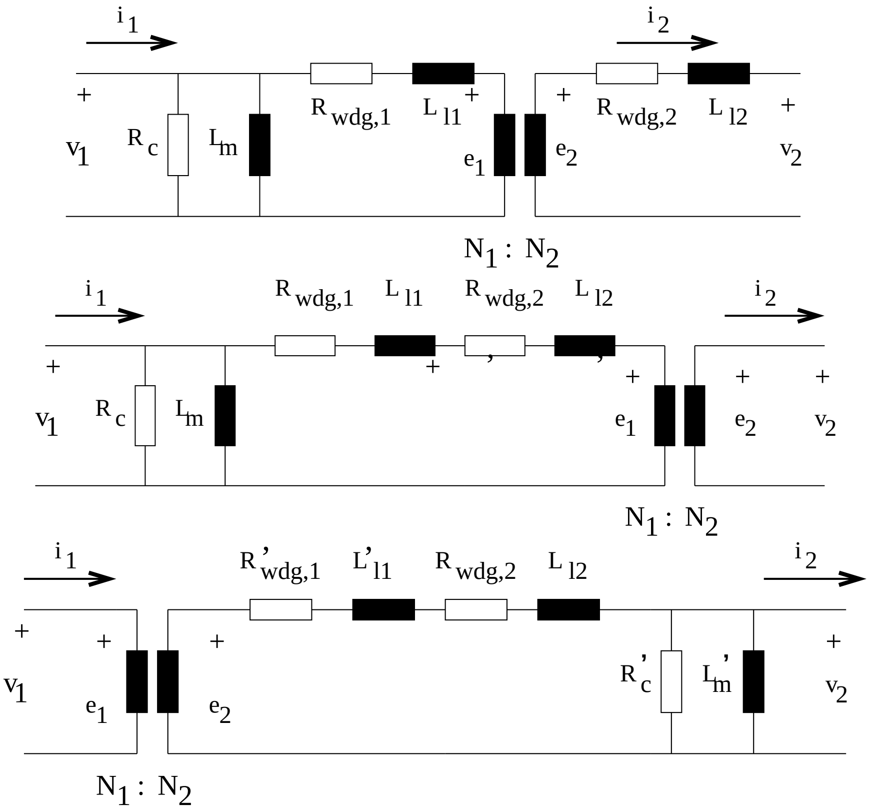


Fig. 3.10 Simplified equivalent circuits of a transformer

The power losses are concentrated in the windings and core:

$$
\begin{aligned}
P_{wdg} &= I_1' R_{wdg} = 110.79W \\
P_{core} &= V_1^2 / R_c = 103.32W \\
P_{loss} &= P_{wdg} + P_{core} = 214.11W \\
\eta = \frac{P_{out}}{P_{in}} = \frac{P_{out}}{(P_{out} + P_{loss})} &= \frac{20kW}{20kW + 221.411W} = 0.9894
\end{aligned}
$$

3.5 PER-UNIT SYSTEM

The idea behind the per unit system is quite simple. We define a base system of quantities, express everything as a percentage (actually per unit) of these quantities, and use all the power and circuit equations with these per unit quantities. In the process the ideal transformer in 3.10 disappears.

Working in p.u. has a some other advantages, e.g. the range of values of parameters is almost the same for small and big transformers.

Working in the per unit system adds steps to the solution process, so one hopes that it simplifies the solution more than it complicates it. At first attempt, the per unit system makes no sense. Let us look at an example:

3.5.1 Example

A load has impedance $10 + j5\Omega$ and is fed by a voltage of $100V$. Calculate the current and power at the load.

Solution 1 *the current will be*

$$\mathbf{I_L} = \frac{\mathbf{V_L}}{\mathbf{Z_L}} = \frac{100}{10 + j5} = 8.94^{\angle -26.57^0} A$$

and the power will be

$$P_L = V_L I_L \cdot pf = 100 \cdot 8.94 \cdot \cos(26.57) = 800W$$

Solution 2 *Let's use the per unit system.*

1. *define a consistent system of values for base. Let us choose $V_b = 50V$, $I_b = 10A$. This means that $Z_b = V_b / I_b = 5\Omega$, and $P_b = V_b \cdot I_b = 500W$, $Q_b = 500VAr$, $S_b = 500VA$.*

2. *Convert everything to pu. $V_{L,pu} = V_L / B_b = 2pu$, $\mathbf{Z}_{L,pu} = (10 + j5)/5 = 2 + j1\ pu$.*

3. *solve in the pu system.*

$$\mathbf{I_{L,pu}} = \frac{\mathbf{V_{L,pu}}}{\mathbf{Z_{L,pu}}} = \frac{2}{2 + j1} = 0.894^{\angle -26.57^0}\ pu$$

$$P_{L,pu} = V_{L_p u} I_{L,pu} \cdot pf = 2 \cdot 0.894 \cdot \cos(26.57^0) = 1.6\ pu$$

4. *Convert back to the SI system*

$$I_L = I_{L,pu} \cdot I_b = 0.894 \cdot 10 = 8.94A$$
$$P_l = P_{L,pu} \cdot P_b = 1.6 \cdot 500 = 800W$$

The second solution was a bit longer and appears to not be worth the effort. Let us now apply this method to a transformer, but be shrewder in choosing our bases. Here we'll need a base system for each side of the ideal transformer, but in order for them to be consistent, the ratio of the voltage and current bases should satisfy:

$$\frac{V_{1b}}{V_{2b}} = \frac{N_1}{N_2} \tag{3.21}$$

$$\frac{I_{1b}}{I_{2b}} = \frac{N_2}{N_1} \tag{3.22}$$

$$\Rightarrow S_{1b} = V_{1b} I_{1b} = V_{2b} I_{2b} = S_{2b} \tag{3.23}$$

i.e. the two base apparent powers are the same, as are the two base real and reactive powers.

We often choose as bases the rated quantities of the transformer on each side, This is convenient, since the transformer most of the time operates at rated voltage (making the pu voltage unity), and the currents and power are seldom above rated, above $1pu$.

Notice that the base impedances on the two sides are related:

$$Z_{1,b} = \frac{V_{1,b}}{I_{1,b}} \tag{3.24}$$

$$Z_{2,b} = \frac{V_{2,b}}{I_{2,b}} = \left(\frac{N_2}{N_1}\right)^2 \frac{V_{1,b}}{I_{1,b}} \tag{3.25}$$

$$Z_{2,b} = \left(\frac{N_2}{N_1}\right)^2 Z_{1,b} \tag{3.26}$$

We notice that as we move impedances from the one side of the transformer to the other, they get multiplied or divided by the square of the turns ratio, $\left(\frac{N_2}{N_1}\right)^2$, but so does the base impedance, hence the pu value of an impedance stays the same, regardless on which side it is.

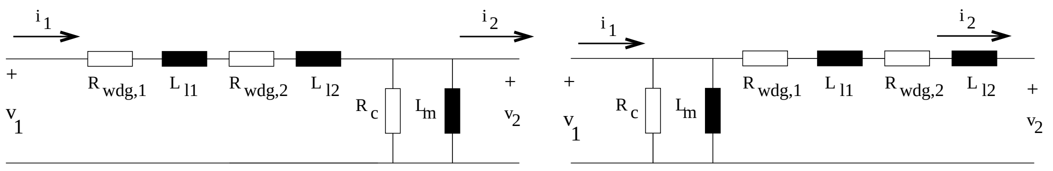
Also we notice, that since the ratio of the voltages of the ideal transformer is $\mathbf{E_1}/\mathbf{E_2} = N_1/N_2$, is equal to the ratio of the current bases on the two sides on the ideal transformer, then

$$\mathbf{E_{1,pu}} = \mathbf{E_{2,pu}}$$

and similarly,

$$\mathbf{I_{1,pu}} = \mathbf{I_{2,pu}}$$

This observation leads to an ideal transformer where the voltages and currents on one side are identical to the voltages and currents on the other side, i.e. the elimination of the ideal transformer, and the equivalent circuits of fig. 3.11 a, b. Let us solve again the same problem as before, with



(a) (b)

Fig. 3.11 Equivalent circuits of a transformer in pu

some added information:

3.5.2 Example

A transformer is rated $30kVA$*,* $4000V/120V$*, with* $R_{wdg,1} = 1.6\Omega$*,* $R_{wdg,2} = 1.44m\Omega$*,* $L_{l1} = 21mH$*,* $L_{l2} = 19\mu H$*,* $R_c = 160k\Omega$*,* $L_m = 450H$*. The voltage at the low voltage side is* $60Hz$*,* $V_2 = 120V$*, and the power there is* $P_2 = 20kW$*, at* $pf = 0.85$ lagging*. Calculate the voltage at the high voltage side and the efficiency of the transformer.*

1. *First calculate the impedances of the equivalent circuit:*

$$\begin{aligned}
V_{1b} &= 4000V \\
S_{1b} &= 30kVA \\
I_{1b} &= \frac{30 \cdot 10^3}{4 \cdot 10^3} = 7.5A \\
Z_{1b} &= \frac{V_{1b}^2}{S_{1b}} = 533\Omega \\
V_{2b} &= 120V \\
S_{2b} &= S_{1b} = 30kVA \\
I_{2b} &= \frac{S_{2b}}{V_{2b}} = 250A \\
Z_{2b} &= \frac{V_{2b}}{I_{2b}} = 0.48\Omega
\end{aligned}$$

2. *Convert everything to per unit: First the parameters:*

$$
\begin{aligned}
R_{wdg,1,pu} &= R_{wdg,1}/Z_{1b} = 0.003\ pu \\
R_{wdg,2,pu} &= R_{wdg,2}/Z_{2b} = 0.003\ pu \\
X_{l1,pu} &= \frac{2\pi 60 L_{l1}}{Z_{1b}} = 0.015\ pu \\
X_{l2,pu} &= \frac{2\pi 60 L_{l2}}{Z_{2b}} = 0.0149\ pu \\
R_{c,pu} &= \frac{R_c}{Z_{1b}} = 300\ pu \\
X_{m,pu} &= \frac{2\pi 60 L_{lm}}{Z_{1b}} = 318 pu
\end{aligned}
$$

Then the load:

$$
\begin{aligned}
V_{2,pu} &= \frac{V_2}{V_{2b}} = 1pu \\
P_{2,pu} &= \frac{P_2}{S_{2b}} = 0.6667pu
\end{aligned}
$$

3. *Solve in the pu system. We'll drop the pu symbol from the parameters and variables:*

$$
\begin{aligned}
\mathbf{I}_2 &= \left(\frac{P_2}{V_2 \cdot pf}\right)^{\angle arccos(pf)} = 0.666 - j0.413pu \\
\mathbf{V}_1 &= \mathbf{V}_2 + \mathbf{I}\left[R_{wdg,1} + R_{wdg,2} + j(X_{l1} + X_{l2})\right] = 1.0172 + j0.0188pu \\
\mathbf{I}_m &= \frac{\mathbf{V}_1}{R_c} + \frac{\mathbf{V}_1}{jX_m} = 0.0034 - j0.0031pu \\
\mathbf{I}_1 &= \mathbf{I}_m + \mathbf{I}_2 = 0.06701 - j0.416\ pu \\
P_{wdg} &= I_2^2\left(R_{wdg,1} + R_{ewg,2}\right) = 0.0037\ pu \\
P_{core} &= \frac{V_1^2}{R_c} = 0.0034pu \\
\eta &= \frac{P_2}{P_{wdg} + P_{core} + P_2} = 0.9894
\end{aligned}
$$

4. *Convert back to SI. The efficiency, η, is dimensionless, hence it stays the same. The voltage, V_1 is*

$$\mathbf{V}_1 = \mathbf{V}_{1,pu} V_{1b} = 4069^{\angle 1^0} V$$

3.6 TRANSFORMER TESTS

We are usually given a transformer, with its frequency, power and voltage ratings, but without the values of its impedances. It is often important to know these impedances, in order to calculate voltage regulation, efficiency etc., in order to evaluate the transformer (e.g. if we have to choose from many) or to design a system. Here we'll work on finding the equivalent circuit of a transformer, through two tests. We'll use the results of these test in the per-unit system.

First we notice that if the relative values are as described in section 3.4, we cannot separate the values of the primary and secondary resistances and reactances. We will lump $R_{1,wdg}$ and $R_{2,wdg}$

together, as well as X_{l1} and X_{l2}. This will leave four quantities to be determined, R_{wdg}, X_l, R_c and X_m.

3.6.1 Open Circuit Test

We leave one side of the transformer open circuited, while to the other we apply rated voltage (i.e. $V_{oc} = 1pu$) and measure current and power. On the open circuited side of the transformer rated voltage appears, but we just have to be careful not to close the circuit ourselves. The current that flows is primarily determined by the impedances X_m and R_c, and it is much lower than rated. It is reasonable to apply this voltage to the low voltage side, since (with the ratings of the transformer in our example) is it easier to apply $120V$, rather than $4000V$. We will use these two measurements to calculate the values of R_c and X_m.

Dropping the subscript *pu*, using the equivalent circuit of figure 3.11b and neglecting the voltage drop on the horizontal part of the circuit, we calculate:

$$P_{oc} = \frac{{V_{oc}}^2}{R_c} = \frac{1}{R_c} \quad (3.27)$$

$$\mathbf{I_{oc}} = \frac{V_{oc}}{R_c} + \frac{V_{oc}}{jX_m}$$

$$I_{oc} = 1\sqrt{\frac{1}{{R_c}^2} + \frac{1}{{X_m}^2}} \quad (3.28)$$

Equations 3.27 and 3.28, allow us to use the results of the short circuit test to calculate the vertical (core) branch of the transformer equivalent circuit.

3.6.2 Short Circuit Test

To calculate the remaining part of the equivalent circuit, i.e the values of R_{wdg} and X_l, we short circuit one side of the transformer and apply rated current to the other. We measure the voltage of that side and the power drawn. On the other side, (the short-circuited one) the voltage is of course zero, but the current is rated. We often apply voltage to the high voltage side, since a) the applied voltage need not be high and b) the rated current on this side is low.

Using the equivalent circuit of figure 3.11a, we notice that:

$$P_{sc} = I_{sc}^2 R_{wdg} = 1 \cdot R_{wdg} \quad (3.29)$$

$$\mathbf{V_{sc}} = \mathbf{I_{sc}}\left(R_{wdg} + jX_l\right)$$

$$V_{sc} = 1 \cdot \sqrt{R_{wdg}^2 + X_l^2} \quad (3.30)$$

Equations 3.29 and 3.30 can give us the values of the parameters in the horizontal part of the equivalent circuit of a transformer.

3.6.1 Example
A $60Hz$ transformer is rated $30kVA$, $4000V/120V$. The open circuit test, performed with the high voltage side open, gives $P_{oc} = 100W$, $I_{oc} = 1.1455A$. The short circuit test, performed with the low voltage side shorted, gives $P_{sc} = 180W$, $V_{sc} = 129.79V$. Calculate the equivalent circuit of the transformer in per unit.

First define bases:

$$\begin{aligned} V_{1b} &= 4000V \\ S_{1b} &= 30kVA \\ I_{1b} &= \frac{30 \cdot 10^3}{4 \cdot 10^3} = 7.5A \\ Z_{1b} &= \frac{V_{1b}^2}{S_{1b}} = 533\Omega \\ V_{2b} &= 120V \\ S_{2b} &= S_{1b} = 30kVA \\ I_{2b} &= \frac{S_{1b}}{V_{2b}} = 250A \\ Z_{2b} &= \frac{V_{1b}}{I_{1b}} = 0.48\Omega \end{aligned}$$

Convert now everything to per unit:

$$\begin{aligned} P_{sc,pu} &= \frac{180}{30 \cdot 10^3} = 0.006ppu \\ V_{sc,pu} &= \frac{129.79}{4000} = 0.0324pu \\ P_{oc,pu} &= \frac{100}{30 \cdot 10^3} = 0.003333pu \\ I_{oc,pu} &= \frac{1.1455}{250} = 0.0046pu \end{aligned}$$

Let's calculate now, dropping the pu subscript:

$$\begin{aligned} P_{sc} &= I_{sc}^2 R_{wdg} \Rightarrow R_{wdg} = P_{sc}/I_{sc}^2 = 1 \cdot P_{sc} = 0.006pu \\ |\mathbf{V_{sc}}| &= |\mathbf{I_{sc}}| \cdot |R_{wdg} + jX_l| = 1 \cdot \sqrt{R_{wdg}^2 + X_l^2} \Rightarrow X_l = \sqrt{V_{sc}^2 - R_{wdg}^2} = 0.0318pu \\ P_{oc} &= \frac{V_{oc}^2}{R_c} \Rightarrow R_c = \frac{1^2}{P_{oc}} = 300pu \\ |\mathbf{I_{oc}}| &= \left| \frac{\mathbf{V_{oc}}}{R_c} + \frac{\mathbf{V_{oc}}}{jX_m} \right| = \sqrt{\frac{1}{R_c^2} + \frac{1}{X_m^2}} \Rightarrow X_m = \frac{1}{\sqrt{I_{oc}^2 - \frac{1}{R_c^2}}} = 318pu \end{aligned}$$

A more typical problem is of the type:

3.6.2 Example

A $60Hz$ transformer is rated $30kVA$, $4000V/120V$. Its short circuit impedance is $0.0324pu$ and the open circuit current is $0.0046pu$. The rated iron losses are $100W$ and the rated winding losses are $180W$. Calculate the efficiency and the necessary primary voltage when the load at the secondary is at rated voltage, $20kW$ at $0.8pf$ lagging.

Working in pu:

$$\begin{aligned}
Z_{sc} &= 0.0324pu \\
P_{sc} &= R_{wdg} = \frac{180}{30 \cdot 10^3} = 6 \cdot 10^{-3} pu \\
\Rightarrow X_l &= \sqrt{Z_{sc}^2 - R_{wdg}^2} = 0.017pu \\
P_{oc} &= \frac{1}{R_c} \Rightarrow R_c = \frac{1}{P_{oc}} = \frac{1}{100/30 \cdot 10^3} = 300pu \\
I_{oc} &= \sqrt{\frac{1}{R_c^2} + \frac{1}{X_m^2}} \Rightarrow X_m = 1 \Bigg/ \sqrt{I_{oc}^2 - \frac{1}{R_c^2}} = 318pu
\end{aligned}$$

Having finished with the transformer data, let us work with the load and circuit. The load power is $20kW$*, hence:*

$$P_2 = \frac{20 \cdot 10^3}{30 \cdot 10^3} = 0.6667pu$$

but the power at the load is:

$$P_2 = V_2 I_2 pf \Rightarrow 0.6667 = 1 \cdot I_2 \cdot 0.8 \Rightarrow I_2 = 0.8333pu$$

Then to solve the circuit, we work with phasors. We use the voltage V_2 *as reference:*

$$\begin{aligned}
\mathbf{V_2} &= V_2 = 1pu \\
\mathbf{I_2} &= 0.8333^{\angle cos^{-1} 0.8} = 0.6667 - j0.5pu \\
\mathbf{V_1} &= \mathbf{V_2} + \mathbf{I_2}\left(R_{wdg} + jX_l\right) = 1.0199 + j0.00183pu \Rightarrow V_1 = 1.02pu \\
P_{wdg} &= I_2^2 \cdot R_{wdg} = 0.0062pu \\
P_c &= V_1^2 / R_c = 0.034pu \\
\eta &= \frac{P_2}{P_2 + P_{wdg} + P_c} = 0.986
\end{aligned}$$

Finally, we convert the voltage to SI

$$V_1 = V_{1,pu} \cdot V_{b1} = 1.021 \cdot 4000 = 4080V$$

3.7 THREE-PHASE TRANSFORMERS

We'll study now three-phase transformers, considering as consisting of three identical one-phase transformers. This method is accurate as far as equivalent circuits and two-port models are our interest, but it does not give us insight into the magnetic circuit of the three-phase transformer. The primaries and the secondaries of the one-phase transformers can be connected either in Δ or in Y. In either case, the rated power of the three-phase transformer is three times that of the one-phase transformers. For Δ connection,

$$V_{ll} = V_{1\phi} \tag{3.31}$$

$$I_l = \sqrt{3} I_{1\phi} \tag{3.32}$$

For Y connection

$$V_{ll} = \sqrt{3} V_{1\phi} \tag{3.33}$$

$$I_l = I_{1\phi} \tag{3.34}$$

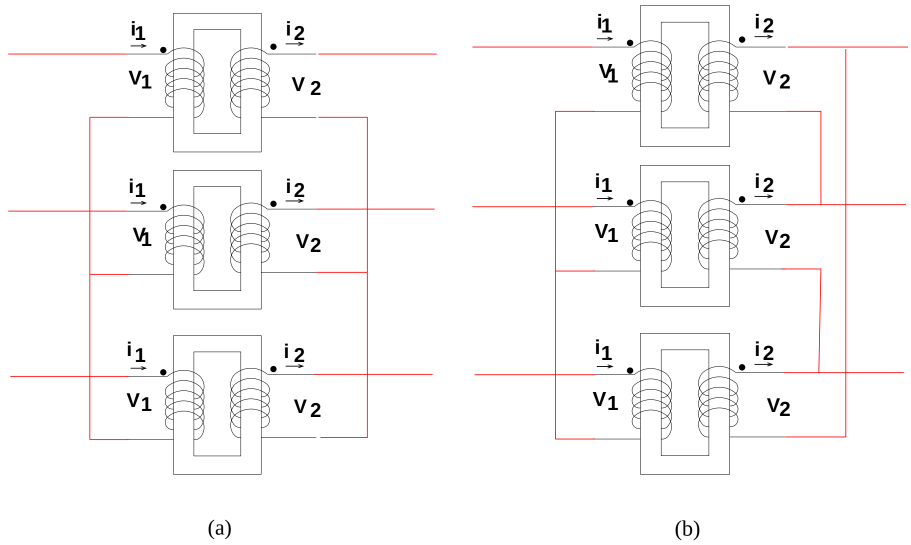


Fig. 3.12 $Y - Y$ and $Y - \Delta$ Connections of three-phase Transformers

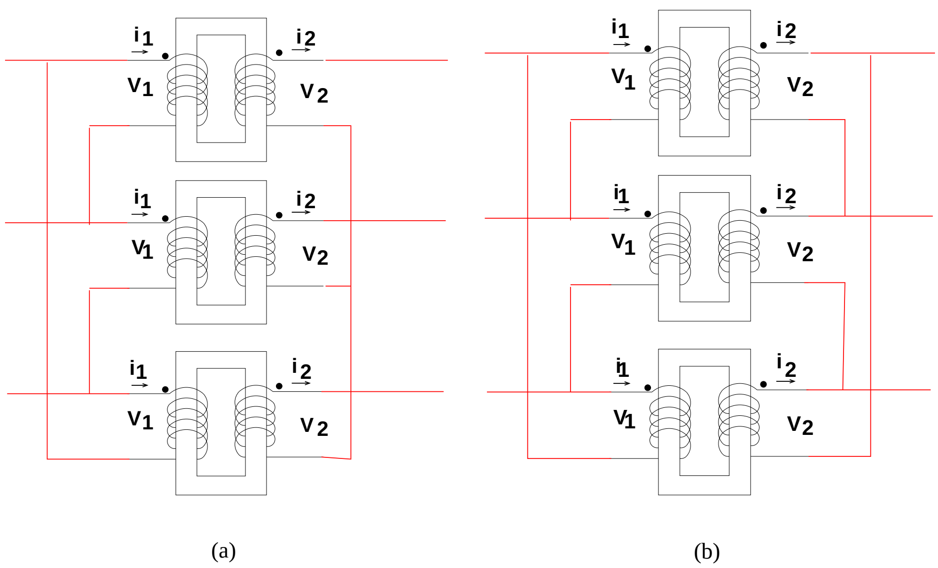


Fig. 3.13 $\Delta - Y$ and $\Delta - \Delta$ Connections of three-phase Transformers

3.8 AUTOTRANSFORMERS

An autotransformer is a transformer where the two windings (of turns N_1 and N_2) are not isolated from each other, but rather connected as shown in figure 3.14. It is clear form this figure that the voltage ratio in an autotransformer is

$$\frac{V_1}{V_2} = \frac{N_1 + N_2}{N_2} \tag{3.35}$$

while the current ratio is

$$\frac{I_2}{I_1} = \frac{N_1 + N_2}{N_2} \tag{3.36}$$

The interesting part is that the coil of turns of N_1 carries current I_1, while the coil of turns N_2 carries the (vectorial) sum of the two currents, $\mathbf{I_1} - \mathbf{I_2}$. So if the voltage ratio where 1, no current would flow through that coil. This characteristic leads to a significant reduction in size of an autotransformer compared to a similarly rated transformer, especially if the primary and secondary voltages are of the same order of magnitude. These savings come at a serious disadvantage, the loss of isolation between the two sides.

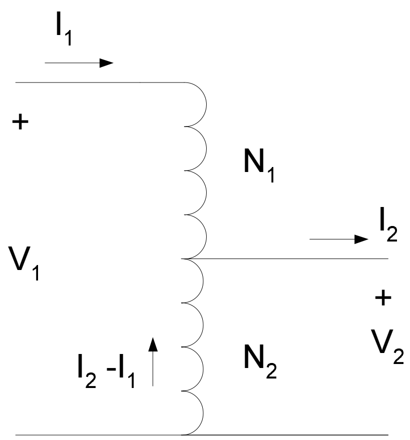


Fig. 3.14 An Autotransformer

Notes

- To understand the operation of transformers we have to use both the Biot-Savart Law and Faraday's law.
- Most transformers operate under or near rated voltage. As the voltage drop in the leakage inductance and winding resistances are small, the iron losses under such operation transformer are close to rated.
- The open- and short-circuit test are just that, tests. They provide the parameters that define the operation of the transformer.
- Three-phase transformers can be considered to be made of three single-phase transformers for the purposes of these notes. The main issue then is to calculate the ratings and the voltages and currents of each.

- Autotransformers are used mostly to vary the voltage a little. It is seldom that an autotransformer will have a voltage ratio greater than two.

4

Concepts of Electrical Machines; DC motors

DC machines have faded from use due to their relatively high cost and increased maintenance requirements. Nevertheless, they remain good examples for electromechanical systems used for control. We'll study DC machines here, at a conceptual level, for two reasons:

1. DC machines although complex in construction, can be useful in establishing the concepts of emf and torque development, and are described by simple equations.

2. The magnetic fields in them, along with the voltage and torque equations can be used easily to develop the ideas of field orientation.

In doing so we will develop basic steady-state equations, again starting from fundamentals of the electromagnetic field. We are going to see the same equations in 'Brushless DC' motors, when we discuss synchronous AC machines.

4.1 GEOMETRY, FIELDS, VOLTAGES, AND CURRENTS

Let us start with the geometry shown in figure 4.1

This geometry describes an outer iron window (stator), through which (i.e. its center part) a uniform magnetic flux is established, say $\hat{\Phi}$. How this is done (a current in a coil, or a permanent magnet) is not important here.

In the center part of the window there is an iron cylinder (called rotor), free to rotate around its axis. A coil of one turn is wound diametrically around the cylinder, parallel to its axis. As the cylinder and its coil rotate, the flux through the coil changes. Figure 4.2 shows consecutive locations of the rotor and we can see that the flux through the coil changes both in value and direction. The top graph of figure 4.3 shows how the flux linkages of the coil through the coil would change, if the rotor were to rotate at a constant angular velocity, ω.

$$\lambda = \hat{\Phi}\cos[\omega t] \tag{4.1}$$

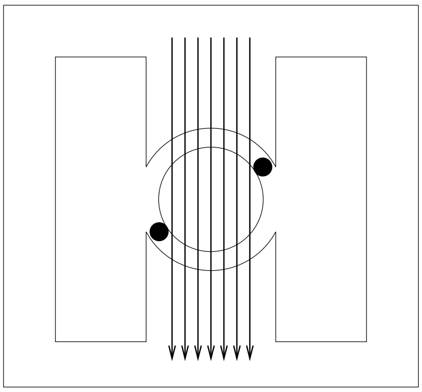


Fig. 4.1 Geometry of an elementary DC motor

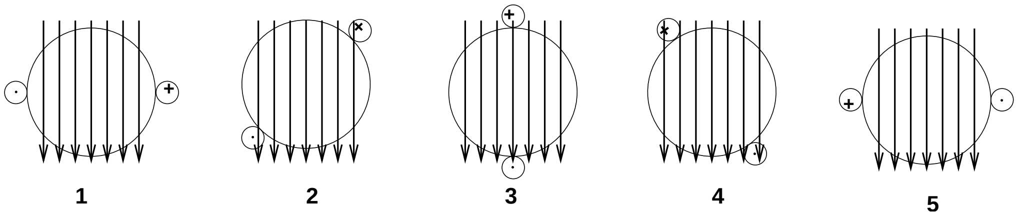


Fig. 4.2 Flux through a coil of a rotating DC machine

Since the flux linking the coil changes with time, then a voltage will be induced in this coil, v_{coil},

$$v_{coil} = \frac{d\lambda}{dt} = -\hat{\Phi}\omega \sin(\omega t) \tag{4.2}$$

shown in the second graph of figure 4.3. The points marked there correspond to the position of the rotor in figure4.2.

This alternating voltage has to somehow be rectified, since this is a *DC* machine. Although this can be done electronically, a very old mechanical method exists. The coil is connected not to the DC source or load, but to two ring segments, solidly attached to it and the rotor, and hence rotating with it. Two 'brushes', i.e. conducting pieces of material (often carbon/copper) are stationary and sliding on these ring segments as shown in figure 4.4

The structure of the ring segments is called a commutator. As it rotates, the brushes make contact with the opposite segments just as the induced voltage goes through zero and switches sign.

Figure 4.5 shows the induced voltage and the terminal voltage seen at the voltmeter of figure 4.4. If a number of coils are placed on the rotor, as shown in figure 4.6, each connected to a commutator segment, the total induced voltage to the coils, E will be:

$$E = k\hat{\Phi}\omega \tag{4.3}$$

where k is proportional to the number of coils.

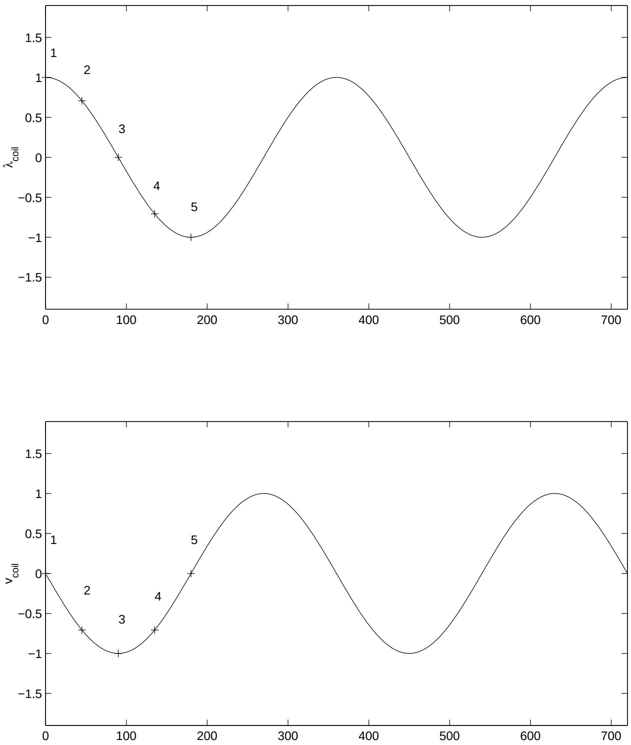


Fig. 4.3 Flux and voltage in a coil of the DC machine in 4.2. Points 1 – 5 represent the coil positions.

Going back to equation 2.25,

$$E \cdot i = T\omega \tag{4.4}$$
$$k\hat{\Phi}\omega i = T\omega \tag{4.5}$$
$$T = k\hat{\Phi} i \tag{4.6}$$

If the electrical machine is connected to a load or a source as in figure4.7, the induced voltage and terminal voltage will be related by:

$$V_{term} = E - i_g R_{wdg} \qquad for\ a\ generator \tag{4.7}$$
$$V_{term} = E + i_m R_{wdg} \qquad for\ a\ motor \tag{4.8}$$

4.1.1 Example

A DC motor, when connected to a $100V$ source and to no load runs at 1200rpm. Its stator resistance is 2Ω. What should be the torque and current if it is fed from a $220V$ supply and its speed is $1500rpm$? Assume that the field is constant.

The first piece of information gives us the constant k. Since at no load the torque is zero and $T = k\Phi i = Ki$, then the current is zero as well. This means that for this operation:

$$V = E = k\Phi\omega = K\omega$$

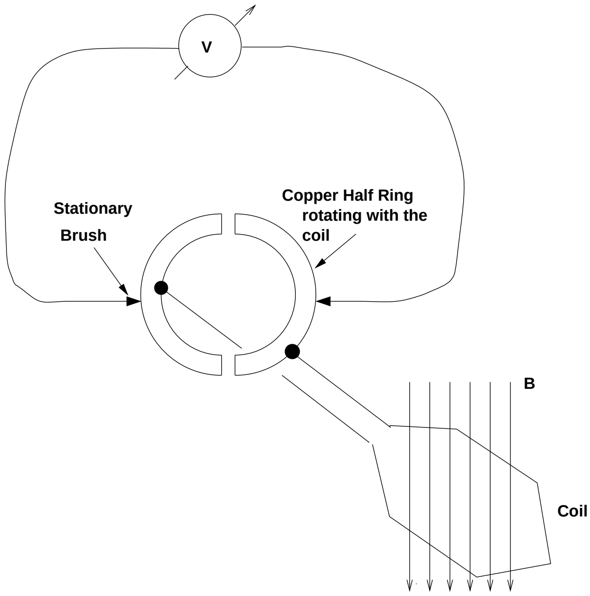


Fig. 4.4 A coil of a DC motor and a commutator with brushes

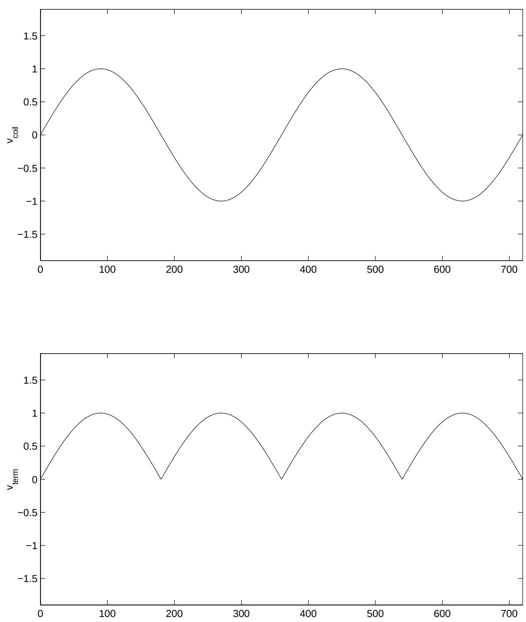


Fig. 4.5 Induced voltage in a coil and terminal voltage in an elementary DC machine

but ω is 1200 rpm, or in SI units:

$$\omega = 1200\frac{2\pi}{60} = 125.66 rad/s$$

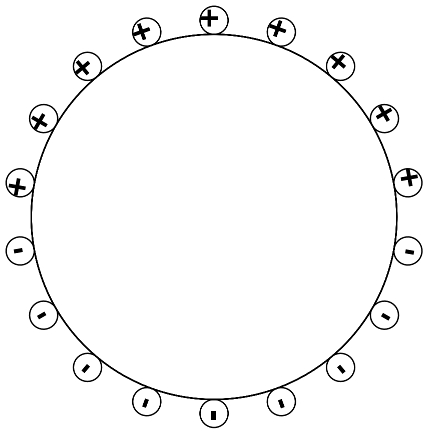


Fig. 4.6 Coils on the rotor of DC machine

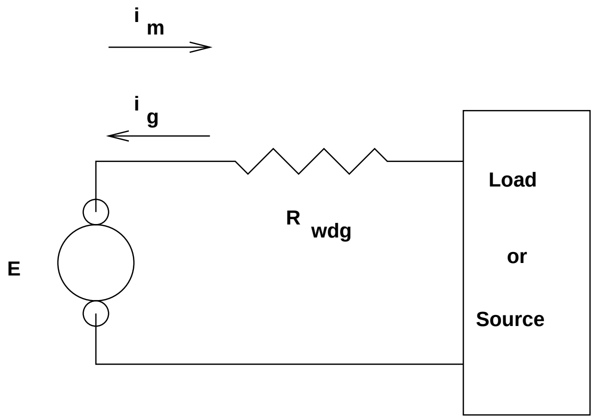


Fig. 4.7 Circuit with a DC machine

And

$$100V = K \cdot 125.66 \Rightarrow K = 0.796Vs$$

At the operating point of interest:

$$\omega_o = 1500rpm = 1500\frac{2\pi}{60} = 157.08rad/s \Rightarrow E = K\omega = 125V$$

For a motor:

$$\begin{aligned} V &= E + IR \\ \Rightarrow 220 &= 125 + I \cdot 2\Omega \\ \Rightarrow I &= 47.5A \\ \Rightarrow T &= KI = 37.81Nm \end{aligned}$$

Notes

- The field of the DC motor can be created either by a DC current or a permanent magnet.
- These two fields, the one coming from the stator and the one coming from the moving rotor, are both stationary (despite rotation) and perpendicular to each other.
- if the direction of current in the stator and in the rotor reverse together, torque will remain in the same direction. Hence if the same current flows in both windings, it could be AC and the motor will not reverse (e.g. hairdryers, power drills).

5

Three-phase Windings

Understanding the geometry and operation of windings in AC machines is essential in understanding how these machines operate. We introduce here the concept of Space Vectors, (or Space Phasors) in its general form, and we see how they are applied to three-phase windings.

5.1 CURRENT SPACE VECTORS

Let us assume that in a uniformly permeable space we have placed three identical windings as shown in figure 5.1. Each carries a time dependent current, $i_1(t)$, $i_2(t)$ and $i_3(t)$. We require that:

$$i_1(t) + i_2(t) + i_3(t) \equiv 0 \tag{5.1}$$

Each current produces a flux in the direction of the coil axis, and if we assume the magnetic medium to be linear, we can find the total flux by adding the individual fluxes. This means that we could produce the same flux by having only one coil, identical to the three, but placed in the direction of the total flux, carrying an appropriate current. Figure 5.2 shows such a set of coils carrying for $i_1 = 5A$, $i_2 = -8A$ and $i_3 - = 3A$ and the resultant coil.

To calculate the direction of the resultant one coil and the current it should carry, all we have to do is create three vectors, each in the direction of one coil, and of amplitude equal to the current of each coil. The sum of these vectors will give the direction of the total flux and hence of the one coil that will replace the three. The amplitude of the vectors will be that of the current of each coil.

Let us assume that the coils are placed at angles 0^0, 120^0 and 240^0. Then their vectorial sum will be:

$$\mathbf{i} = i^{\angle\phi} = i_1 + i_2 e^{j120^0} + i_3 e^{j240^0} \tag{5.2}$$

We call $\mathbf{i}$, defined thus, a space vector, and we notice that if the currents i_1, i_2 and i_3 are functions of time, so will be the amplitude and the angle of $\mathbf{i}$. By projecting the three constituting currents on the horizontal and vertical axis, we can find the real ($i_d = \Re[\mathbf{i}]$) and imaginary ($i_q = \Im[\mathbf{i}]$) parts of

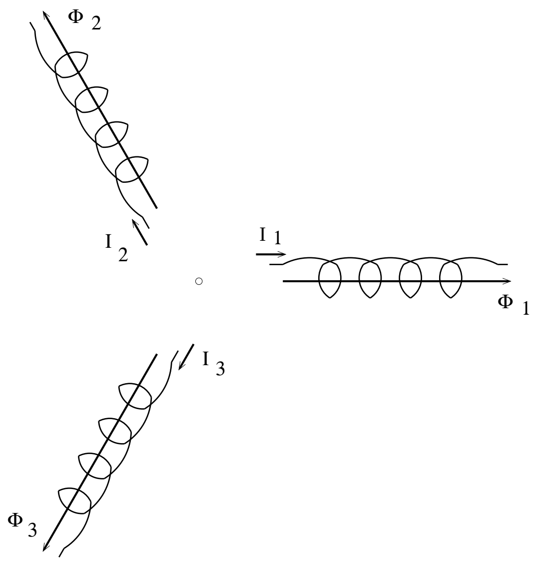
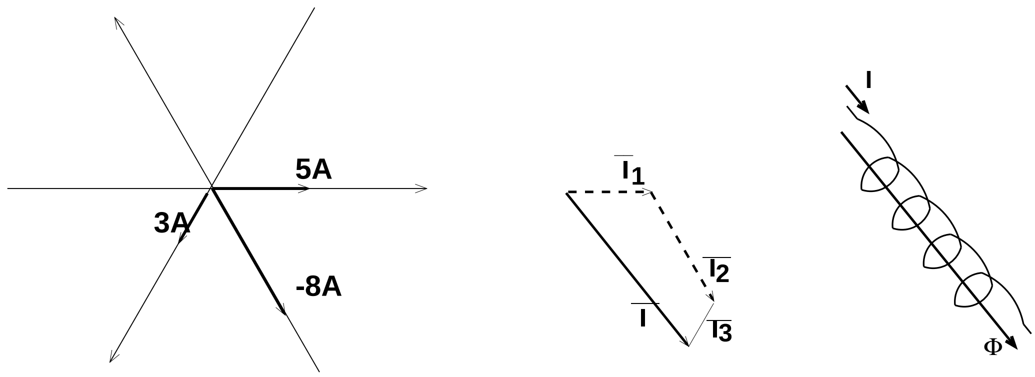


Fig. 5.1 Three phase concentrated windings



(a) (b) (c)

Fig. 5.2 Currents in three windings (a), Resultant space vector (b), and corresponding winding position (c)

it. Also, from the definition of the current space vector we can reconstruct the constituent currents:

$$\begin{aligned} i_1(t) &= \frac{2}{3}\Re[\mathbf{i}(t)] \\ i_2(t) &= \frac{2}{3}\Re[\mathbf{i}(t)e^{-j\gamma}] \\ i_3(t) &= \frac{2}{3}\Re[\mathbf{i}(t)e^{-j2\gamma}] \end{aligned} \tag{5.3}$$

$$\gamma = 120^0 = \frac{2\pi}{3}\, rad \tag{5.4}$$

5.2 STATOR WINDINGS AND RESULTING FLUX DENSITY

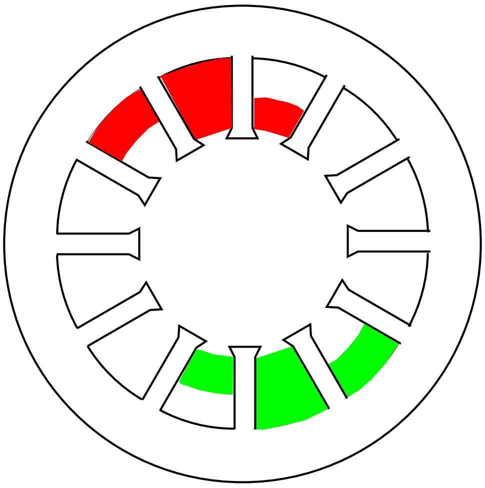


Fig. 5.3 A Stator Lamination

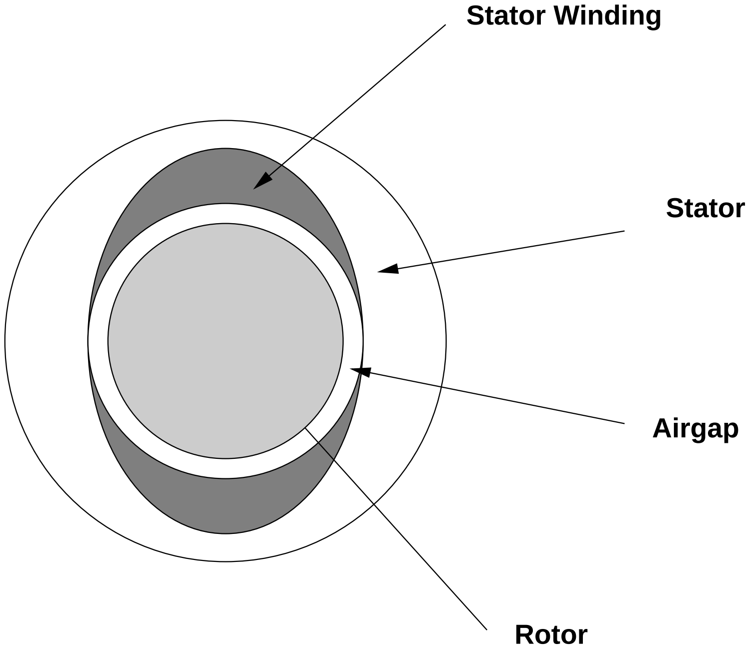


Fig. 5.4 A sinusoidal winding on the stator

Assume now that these windings are placed in a fixed structure, the stator, which is surrounds a rotor. Figure 5.3 shows a typical stator cross-section, but for the present we'll consider the stator as a steel tube. Figure 5.5 shows the windings in such a case. Instead of being concentrated, they are sinusoidally distributed as shown in figure 5.4. Sinusoidal distribution means that the number of turns dN_s covering an angle $d\theta$ at a position θ and divided by $d\theta$ is a sinusoidal function of the angle θ. This turns density, $n_{s1}(\theta)$, is then:

$$\frac{dn_s}{d\theta} = n_{s1}(\theta) = \hat{n}_s \sin\theta$$

and for a total number of turns N_s in the winding:

$$N_s = \int_0^{\pi} n_{s1}(\theta) d\theta \Rightarrow n_{s1}(\theta) = \frac{N_s}{2}\sin\theta$$

We now assign to the winding we are discussing a current i_1. To find the flux density in the airgap

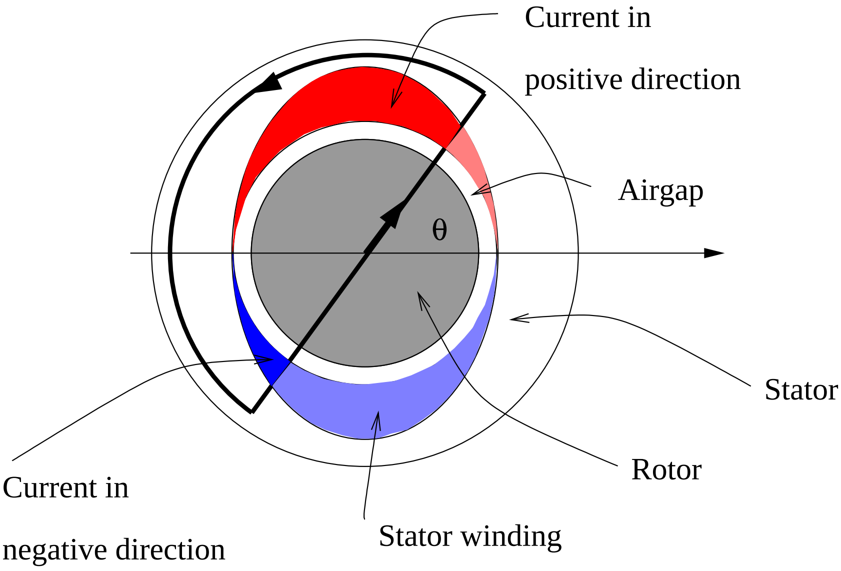


Fig. 5.5 Integration path to calculate flux density in the airgap

between rotor and stator we choose an integration path as shown in figure 5.5. This path is defined by the angle θ and we can notice that because of symmetry the flux density at the two airgap segments in the path is the same. If we assume the permeability of iron to be infinite, then $H_{iron} = 0$ and:

$$\begin{aligned} 2H_{g1}(\theta)g &= \int_{\theta}^{\theta+\pi} i_1 n_{s1}(\phi)d\phi \\ \frac{2B_{g1}(\theta)}{\mu_0}g &= i_1 N_s \cos\theta \\ B_{g1}(\theta) &= i_1 \frac{N_s\mu_0}{2g}\cos\theta \end{aligned} \tag{5.5}$$

This means that for a given current i_1 in the coil, the flux density in the air gap varies sinusoidally with angle, but as shown in figure 5.6 it reaches a maximum at angle $\theta = 0$. For the same machine and conditions as in 5.6, 5.7 shows the plots of turns density, $n_s(\theta)$ and flux density, $B_g(\theta)$ in cartesian coordinates with θ in the horizontal axis.

If the current i_1 were to vary sinusoidally in time, the flux density would also change in time, maintaining its space profile but changing only in amplitude. This would be considered a wave, as it

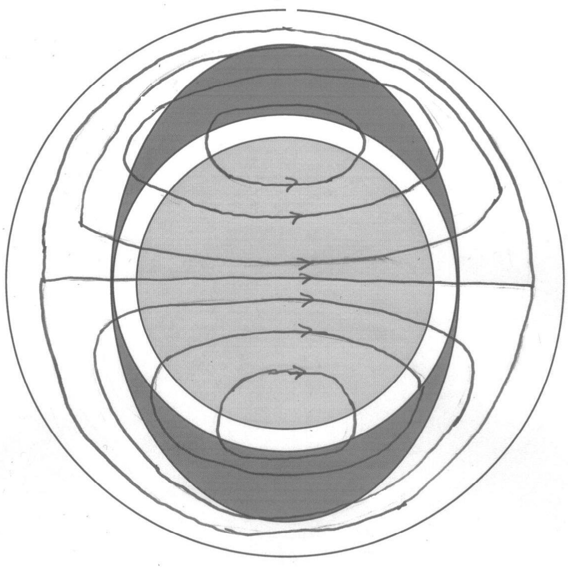


Fig. 5.6 Sketch of the flux in the airgap

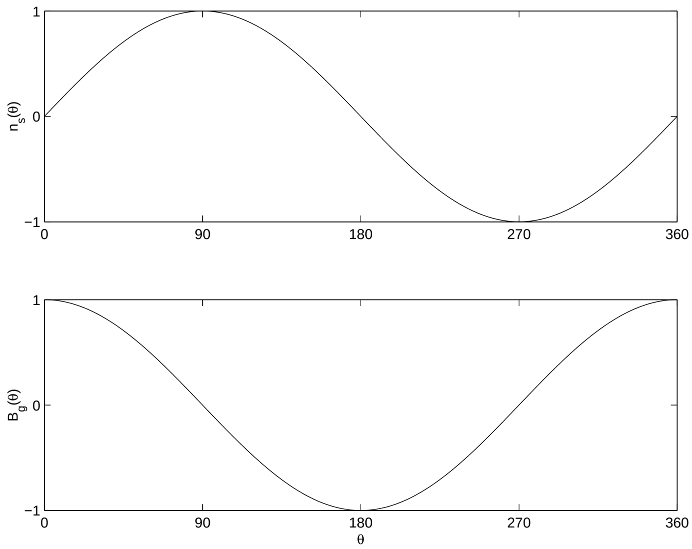


Fig. 5.7 Turns density on the stator and air gap flux density vs. θ

changes in time and space. The nodes of the wave, where the flux density is zero, will remain at 90^0 and 270^0, while the extrema of the flux will remain at 0^0 and 180^0.

Consider now an additional winding, identical to the first, but rotated with respect to it by 120^0. For a current in this winding we'll get a similar airgap flux density as before, but with nodes at $90^0 + 120^0$ and at $270^0 + 120^0$. If a current i_2 is flowing in this winding, then the airgap flux density due to it will follow a form similar to equation 5.5 but rotated $120^0 = \frac{2\pi}{3}$.

$$B_{g2}(\theta) = i_2 \frac{N_s \mu_0}{2g} \cos(\theta - \frac{2\pi}{3}) \tag{5.6}$$

Similarly, a third winding, rotated yet another 120^0 and carrying current i_3, will produce airgap flux density:

$$B_{g3}(\theta) = i_3 \frac{N_s \mu_0}{2g} \cos(\theta - \frac{4\pi}{3}) \tag{5.7}$$

Superimposing these three flux densities, we get yet another sinusoidally distributed airgap flux density, that could equivalently come from a winding placed at an angle ϕ and carrying current i:

$$B_g(\theta) = B_{g1}(\theta) + B_{g2}(\theta) + B_{g3}(\theta) = i \frac{N_s \mu_0}{2g} \cos(\theta + \phi) \tag{5.8}$$

This means that as the currents change, the flux could be due instead to only one sinusoidally distributed winding with the same number of turns. The location, $\phi(t)$, and current, $i(t)$, of this winding can be determined from the current space vector:

$$\mathbf{i}(t) = i(t)^{\angle \phi(t)} = i_1(t) + i_2(t) e^{j120^0} + i_3(t) e^{j240^0}$$

5.2.1 Balanced, Symmetric Three-phase Currents

If the currents i_1, i_2, i_3 form a balanced three-phase system of frequency $f_s = \omega_s / 2\pi$, then we can write:

$$\begin{aligned}
i_1 &= \sqrt{2} I \cos(\omega_s t + \phi_1) = \frac{\sqrt{2}}{2} \left[\mathbf{I}_s e^{j\omega_s t} + \mathbf{I}_s e^{-j\omega_s t} \right] \\
i_2 &= \sqrt{2} I \cos(\omega_s t - \phi_1 + \frac{2\pi}{3}) = \frac{\sqrt{2}}{2} \left[\mathbf{I}_s e^{j(\omega_s t - 2\pi/3)} + \mathbf{I}_s e^{-j(\omega_s t - 2\pi/3)} \right] \\
i_3 &= \sqrt{2} I \cos(\omega_s t - \phi_1 + \frac{4\pi}{3}) = \frac{\sqrt{2}}{2} \left[\mathbf{I}_s e^{j(\omega_s t - 4\pi/3)} + \mathbf{I}_s e^{-j(\omega_s t - 4\pi/3)} \right]
\end{aligned} \tag{5.9}$$

where $\mathbf{I}$ is the phasor corresponding to the current in phase 1. The resultant space vector is

$$\mathbf{i}_s(t) = \frac{3}{2} \frac{\sqrt{2}}{2} \mathbf{I} e^{j\omega_s t} = \frac{3}{2} \frac{\sqrt{2}}{2} I e^{j(\omega_s t + \phi_1)} \qquad \mathbf{I} = I e^{j(\phi_1 + \frac{\pi}{2})} \tag{5.10}$$

The resulting flux density wave is then:

$$B(\theta, t) = \frac{3}{2} \sqrt{2} I \frac{N_s \mu_0}{2g} \cos(\omega_s t + \phi_1 - \theta) \tag{5.11}$$

which shows a travelling wave, with a maximum value $\hat{B} = \frac{3}{2} \sqrt{2} I \frac{N_s}{\mu_0}$. This wave travels around the stator at a constant speed ω_s, as shown in figure 5.8

5.3 PHASORS AND SPACE VECTORS

It is easy at this point to confuse space vectors and phasors. A current phasor, $\mathbf{I} = I e^{j\phi_0}$, describes *one* sinusoidally varying current, of frequency ω, amplitude $\sqrt{2} I$ and initial phase ϕ_0. We can

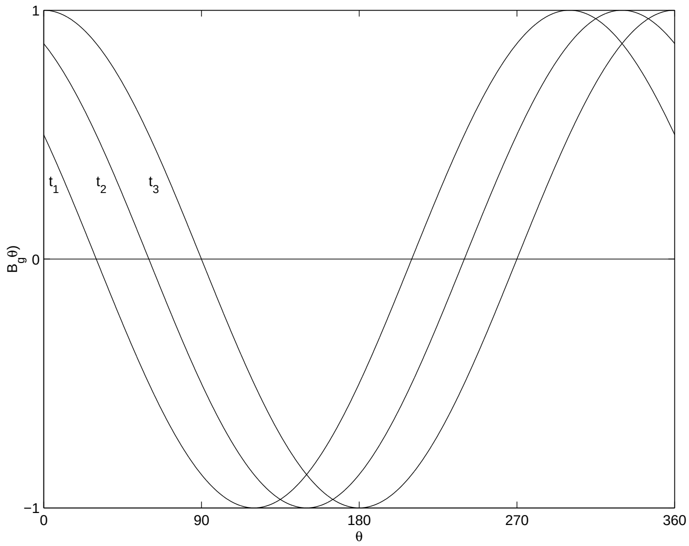


Fig. 5.8 Airgap flux density profile, $t_3 > t_2 > t_1$

reconstruct the sinusoid from the phasor:

$$i(t) = \frac{\sqrt{2}}{2}\left[\mathbf{I}e^{j\omega t} + \mathbf{I}^* e^{-j\omega t}\right] = \sqrt{2}I\cos(\omega t + \phi_0) = \Re\left(\mathbf{I}e^{j\omega t}\right) \tag{5.12}$$

Although rotation is implicit in the definition of the phasor, no rotation is described by it.

On the other hand, the definition of a current space vector requires *three* currents that sum to zero. These currents are implicitly in windings symmetrically placed, but the currents themselves are not necessarily sinusoidal. Generally the amplitude and angle of the space vector changes with time, but no specific pattern is *a priori* defined. We can reconstruct the three currents that constitute the space vector from equation 5.3.

When these constituent currents form a balanced, symmetric system, of frequency ω_s, then the resultant space vector is of constant amplitude, rotating at constant speed. In that case, the relationship between the phasor of one current and the space vector is shown in equation 5.10.

5.3.1 Example

Let us take three balanced sinusoidal currents with amplitude 1*, i.e. rms value of* $1/\sqrt{2}A$*. Choose an initial phase angle such that:*

$$\begin{aligned} i_1(t) &= 1\cos(\omega t)A \\ i_2(t) &= 1\cos(\omega t - 2\pi/3)A \\ i_2(t) &= 1\cos(\omega t - 4\pi/3)A \end{aligned}$$

When $\omega t = 0$*, as shown in figure 5.9a,*

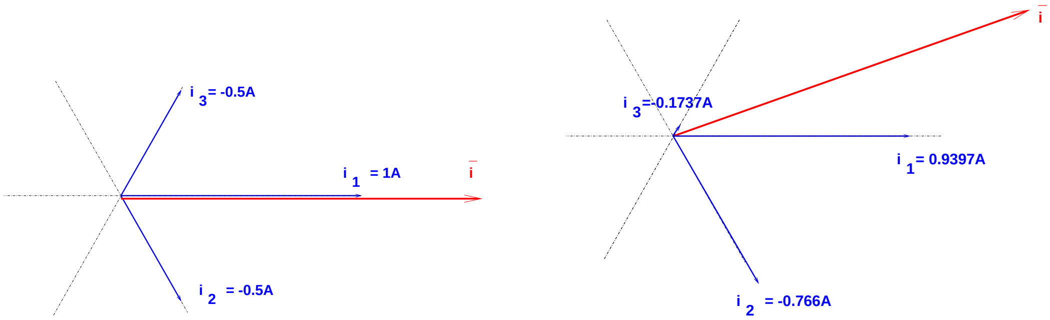
$$\begin{aligned} i_1 &= 1A \\ i_2 &= -0.5A \\ i_3 &= -0.5A \end{aligned}$$

$$\mathbf{i} = i_1 + i_2 e^{j120^0} + i_3 e^{j240^0} = 1.5^{\angle 0} A$$

and later, when $\omega t = 20^0 = \pi/9\,rad$*, as shown in figure 5.9b,*

$$\begin{aligned} i_1 &= 0.939A \\ i_2 &= -0.766A \\ i_3 &= -0.174A \end{aligned}$$

$$\mathbf{i} = i_1 + i_2 e^{j120^0} + i_3 e^{j240^0} = 1.5^{\angle 20^0} A$$



(a) (b)

Fig. 5.9 Space vector movement for sinusoidal, symmetric three-phase currents

5.4 MAGNETIZING CURRENT, FLUX AND VOLTAGE

Let us now see what results this rotating flux has on the windings, using Faraday's law. From this point on we'll use sinusoidal symmetric three-phase quantities.

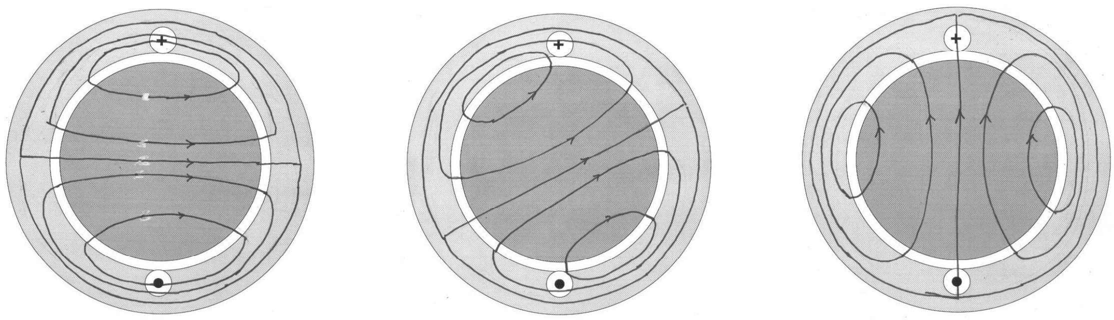
We look again at our three real stationary windings linked by a rotating flux. For example, when the current is maximum in phase 1, the flux is as shown in figure 5.10a, linking all of the turns in phase 1. Later, the flux has rotated as shown in figure 5.10b, resulting in lower flux linkages with the phase 1 windings. When the flux has rotated 90^0, as in 5.10c the flux linkages with the phase 1 winding are zero.

To calculate the flux linkages λ we have to take a turn of the winding, placed at angle θ, as shown in figure 5.11. The flux through this coil is:

$$\Phi(t,\theta) = \int_{\theta-\pi}^{\theta} B_g(t,\phi)dA = lr \int_{\theta-\pi}^{\theta+} B_g(t,\phi)d\phi \tag{5.13}$$

But the number of turns linked by this flux is $dn_s(\theta) = n_s(\theta)d\theta$, so the flux linkages for these turns are:

$$d\lambda = n_s(\theta)d\theta \cdot \Phi(\theta)$$



(a) (b) (c)

Fig. 5.10 Rotating flux and flux linkages with phase 1

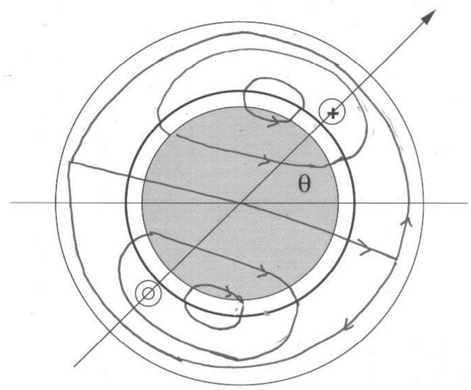


Fig. 5.11 Flux linkages of one turn

To find the flux linkages λ_1 for all of the coil, we have to integrate the flux linkages over all turns of coil 1:

$$\lambda_1 = \int_0^{\pi} \lambda(\theta) d\theta$$

giving us at the end:

$$\lambda_1(t) = \frac{N_s^2 lr}{8g} 3\pi\mu_0 \sqrt{2} I \cos(\omega t + \phi_1) = L_M \sqrt{2} I \cos(\omega t + \phi_1) \tag{5.14}$$

which means that the flux linkages in coil 1 are in phase with the current in this coil and proportional to it. The flux linkages of the other two coils, 2 and 3, are identical to that of coil 1, and lagging in time by 120^0 and 240^0. With these three quantities we can create a flux-linkage space vector, $\boldsymbol{\lambda}$.

$$\boldsymbol{\lambda} \equiv \lambda_1 + \lambda_2 e^{j120^0} + \lambda_3 e^{j240^0} = L_M \, \mathbf{i} \tag{5.15}$$

Since the flux linkages of each coil vary, and in our case sinusoidally, a voltage is induced in each of these coils. The induced voltage in each coil is 90^0 ahead of the current in it, bringing to mind the relationship of current and voltage of an inductor. Notice though, that it is not just the current in the winding that causes the flux linkages and the induced voltages, but rather the current in all three windings. Still, we call the constant L_M magnetizing inductance.

$$\begin{aligned}
e_1(t) &= \frac{d\lambda_1}{dt} = \omega\sqrt{2}I\cos(\omega t + \phi_1 + \frac{\pi}{2}) \\
e_2(t) &= \frac{d\lambda_2}{dt} = \omega\sqrt{2}I\cos(\omega t + \phi_1 + \frac{\pi}{2} - \frac{2\pi}{3}) \\
e_3(t) &= \frac{d\lambda_3}{dt} = \omega\sqrt{2}I\cos(\omega t + \phi_1 + \frac{\pi}{2} - \frac{4\pi}{2})
\end{aligned} \tag{5.16}$$

and of course we can define voltage space vectors $\mathbf{e}$:

$$\mathbf{e} = e_1 + e_2 e^{j120^0} + e_3 e^{j240^0} = j\omega L_M\,\mathbf{i} \tag{5.17}$$

Note that the flux linkage space vector $\boldsymbol{\lambda}$ is aligned with the current space vector, while the voltage space vector $\mathbf{e}$ is ahead of both by 90^0. This agrees with the fact that the individual phase voltages lead the currents by 90^0, as shown in figure 5.12.

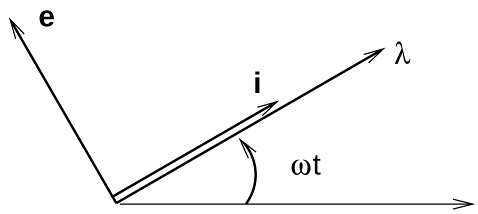


Fig. 5.12 Magnetizing current, flux-linkage and induced voltage space vectors

6

Induction Machines

Induction machines are often described as the 'workhorse of industry'. This clicè reflects the reality of the qualities of these machines. They are cheap to manufacture, rugged and reliable and find their way in most possible applications. Variable speed drives require inexpensive power electronics and computer hardware, and allowed induction machines to become more versatile. In particular, vector or field-oriented control allows induction motors to replace DC motors in many applications

6.1 DESCRIPTION

The stator of an induction machine is a typical three-phase one, as described in the previous chapter. The rotor can be one of two major types. Either a) it is wound in a fashion similar to that of the stator with the terminals led to slip rings on the shaft, as shown in figure 6.1, or b) it is made with shorted

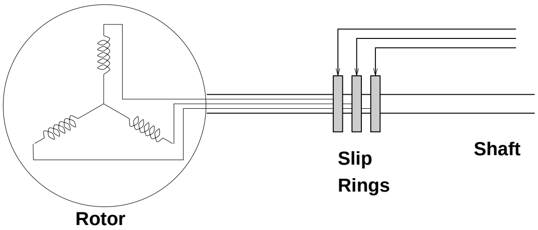


Fig. 6.1 Wound rotor slip rings and connections

bars. Figure 6.2 shows the rotor of such a machine, while figures 6.3 show the shorted bars and the laminations.

The picture of the rotor bars is not easy to obtain, since the bars are formed by casting aluminum in the openings of the rotor laminations. In this case the iron laminations were chemically removed.

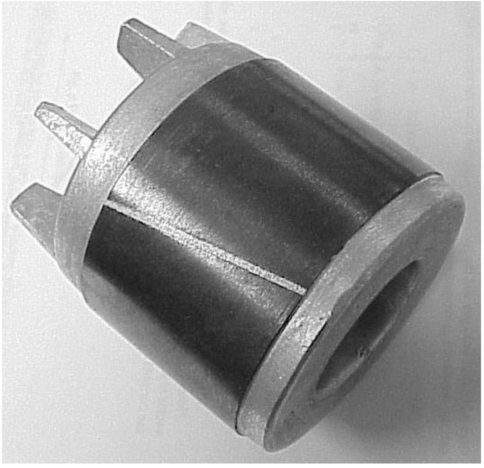
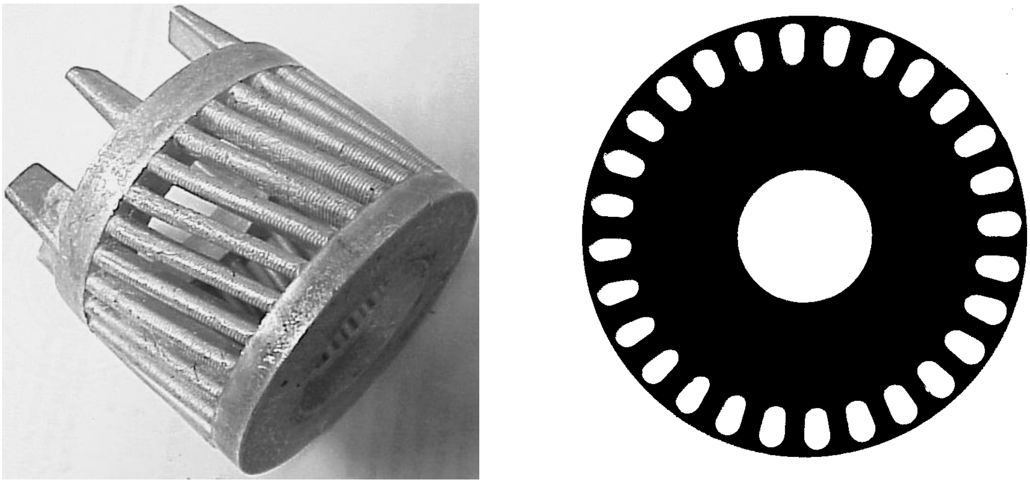


Fig. 6.2 Rotor for squirrel cage induction motor



(a) Rotor bars and rings

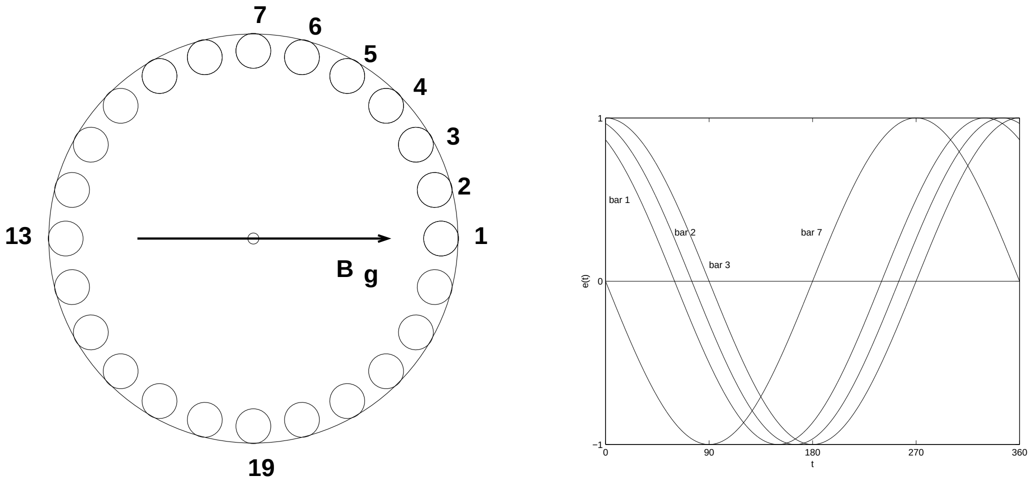
(b) Rotor

Fig. 6.3 Rotor Components of a Squirrel Cage Induction Motor

6.2 CONCEPT OF OPERATION

As these rotor windings or bars rotate within the magnetic field created by the stator magnetizing currents, voltages are induced in them. If the rotor were to stand still, then the induced voltages would be very similar to those induced in the stator windings. In the case of squirrel cage rotor, the voltage induced in the bars will be slightly out of phase with the voltage in the next one, since the flux linkages will change in it after a short delay.

If the rotor is moving at synchronous speed, together with the field, no voltage will be induced in the bars or the windings.



(a) Rotor bars in the stator field

(b) Voltages in rotor bars

Fig. 6.4

Generally when the synchronous speed is $\omega_s = 2\pi f_s$, and the rotor speed ω_0, the frequency of the induced voltages will be f_r, where $2\pi f_r = \omega_s - \omega_0$. Maxwell's equation becomes here:

$$\overrightarrow{\mathcal{E}} = \overrightarrow{v} \times \overrightarrow{B_g} \tag{6.1}$$

where $\overrightarrow{v}$ is the relative velocity of the rotor with respect to the field:

$$v = (\omega_s - \omega_0)r \tag{6.2}$$

Since a voltage is induced in the bars, and these are short-circuited, currents will flow in them. The current density $\overrightarrow{J}(\theta)$ will be:

$$\overrightarrow{J}(\theta) = \frac{1}{\rho}\overrightarrow{\mathcal{E}} \tag{6.3}$$

These currents are out of phase in different bars, just like the induced voltages. To simplify the analysis we can consider the rotor as one winding carrying currents sinusoidally distributed in space. This will be clearly the case for a wound rotor. It will also be the case for uniformly distributed rotor bars, but now each bar, located at an angle θ will carry different current, as shown in figure 6.5 a:

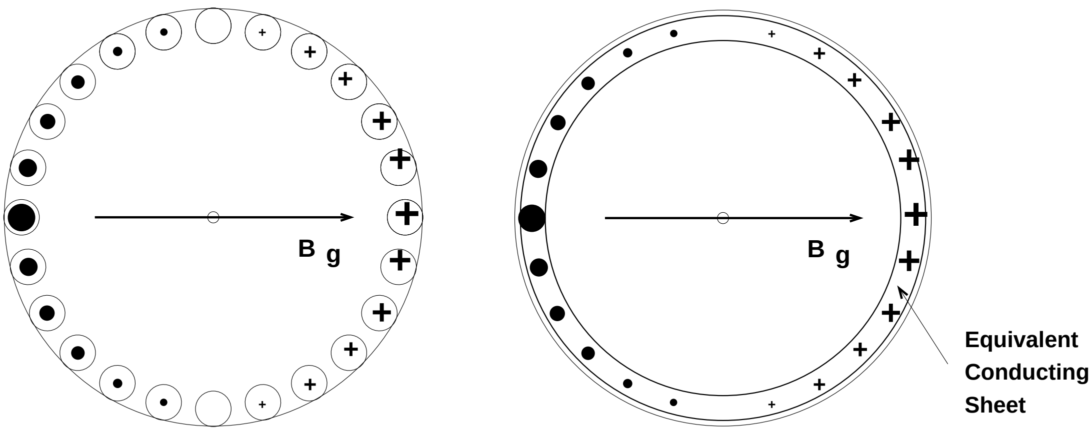
$$\mathbf{J} = \frac{1}{\rho}(\omega_{\mathbf{s}} - \omega_{\mathbf{0}}) \cdot \mathbf{B_g}(\theta) \tag{6.4}$$

$$J(\theta) = \frac{1}{\rho}(\omega_s - \omega_0)\hat{B}_g \sin(\theta) \tag{6.5}$$

We can replace the bars with a conductive cylinder as shown in figure 6.5 b.

We define as slip s the ratio:

$$s = \frac{\omega_s - \omega_0}{\omega_s} \tag{6.6}$$



(a) Currents in rotor bars

(b) Equivalent Current Sheet in Rotor

Fig. 6.5 Current Distribution in equivalent conducting sheet

At starting the speed is zero, hence $s = 1$, and at synchronous speed, $\omega_s = \omega_0$, hence $s = 0$. Above synchronous speed $s < 0$, and when the rotor rotates in a direction opposite of the magnetic field $s > 1$.

6.2.1 Example

The rotor of a two-pole 3-phase induction machine rotates at 3300rpm, while the stator is fed by a three-phase system of voltages, at $60Hz$. What are the possible frequencies of the rotor voltages?

At 3300 rpm

$$\omega_o = 3300\frac{2\pi}{60} = 345.6rad/s \quad \textit{while} \quad \omega_s = 377rad/s$$

These two speeds can be in the opposite or the same direction, hence:

$$\omega_r = \omega_s - \omega_o = 377 \pm 345.6 = 722.58rad/s \quad \textit{or} \quad 31.43rad/s$$
$$f_r = 115Hz \quad \textit{or} \quad = 5Hz$$

6.3 TORQUE DEVELOPMENT

We can now calculate forces and torque on the rotor. We'll use the formulae:

$$F = Bli, \qquad T = F \cdot r \tag{6.7}$$

since the flux density is perpendicular to the current. As l we'll use the length of the conductor, i.e. the depth of the motor. We consider an equivalent thickness of the conducting sheet d_e. A is the cross section of all the bars.

$$A = n_{rotor\ bars}\frac{\pi}{4}d^2 = 2\pi r d_e \tag{6.8}$$

$$d_e = \frac{n_{rotor\ bars}d^2}{8r} \tag{6.9}$$

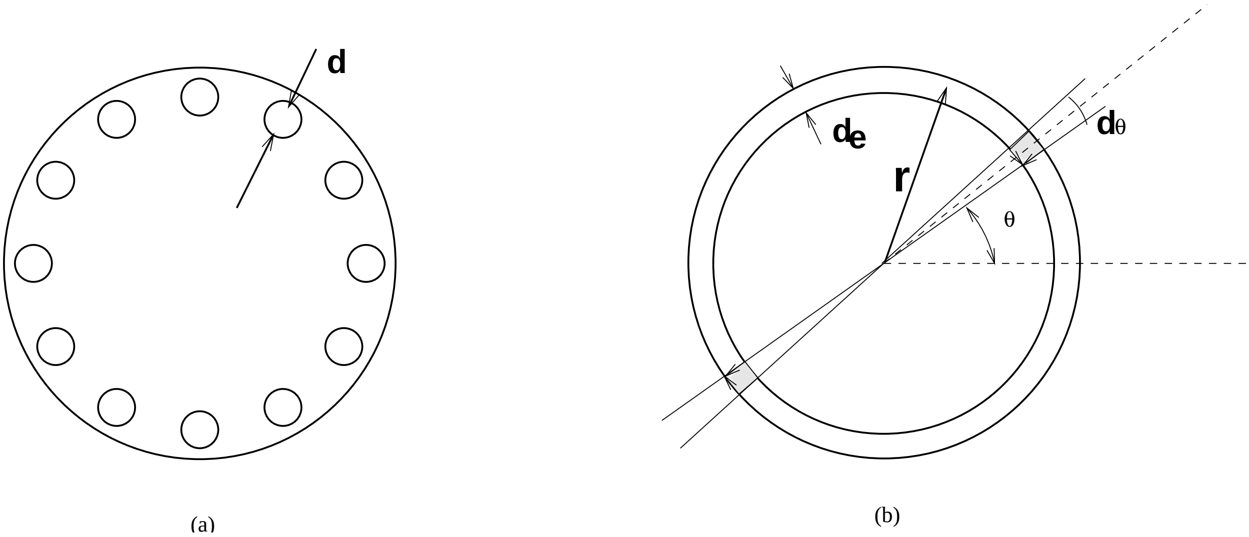


Fig. 6.6 Calculation of Torque

For a small angle $d\theta$ at an angle θ, we calculate the contribution to the total force and torque:

$$\begin{aligned}
dF &= (JdA) \cdot B_g \cdot l, \quad B_g = \hat{B}_g \sin(\theta) \\
dF &= (Jd_e r\, d\theta) B_g \\
dT &= r\, dF \\
T &= \int_{\theta=0}^{\theta=2\pi} dT = \left(\frac{2\pi r^2 l d_e}{\rho} \right) \hat{B}_g^2 (\omega_s - \omega_0) \qquad (6.10)
\end{aligned}$$

Flux density in the airgap is not an easy quantity to work with, so we can use the relationship between flux density (or flux linkages) and rotor voltage and finally get:

$$T = \left(\frac{8}{\pi} \frac{r d_e}{N_s^2 \rho l} \right) \Lambda_s^2 (\omega_s - \omega_0) \qquad \text{where} \qquad \Lambda_s = \left(\frac{E_s}{\omega_s} \right) \qquad (6.11)$$

Although the constants in equations 6.10 and 6.11 are important we should focus more on the variables. We notice that in equation 6.10 the torque is proportional to the frequency of the rotor currents, $(\omega_s - \omega_0)$ and the square of the flux density. This is so since the torque comes from the interaction of the flux density B_g and the rotor currents. But the rotor currents are induced (*induction* motor) due to the flux B_g and the relative speed $\omega_s - \omega_0$.

On the other hand, equation 6.11 gives us torque as a function of more accessible quantities, stator induced voltage E_s and frequency ω_s. This is so, since there is a very simple and direct relationship between stator induced voltage, flux (or flux linkages) and frequency.

6.4 OPERATION OF THE INDUCTION MACHINE NEAR SYNCHRONOUS SPEED

We already determined that the voltages induced in the rotor bars are of slip frequency, $f_r = (\omega_s - \omega_0)/(2\pi)$. At rotor speeds near synchronous, this frequency, f_r is quite small. The rotor bars in a squirrel cage machine possess resistance and leakage inductance, but at very low frequencies, i.e near synchronous speed, we can neglect this inductance. The rotor currents therefore are limited near synchronous speed by the rotor resistance only.

The induced rotor-bar voltages and currents form space vectors. These are perpendicular to the stator magnetizing current and in phase with the space vectors of the voltages induced in the stator as shown in figure 6.7 and figure

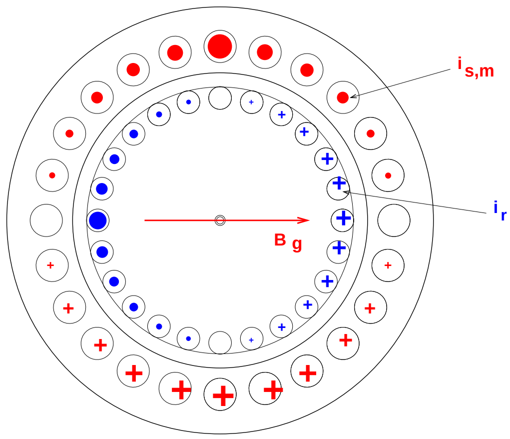
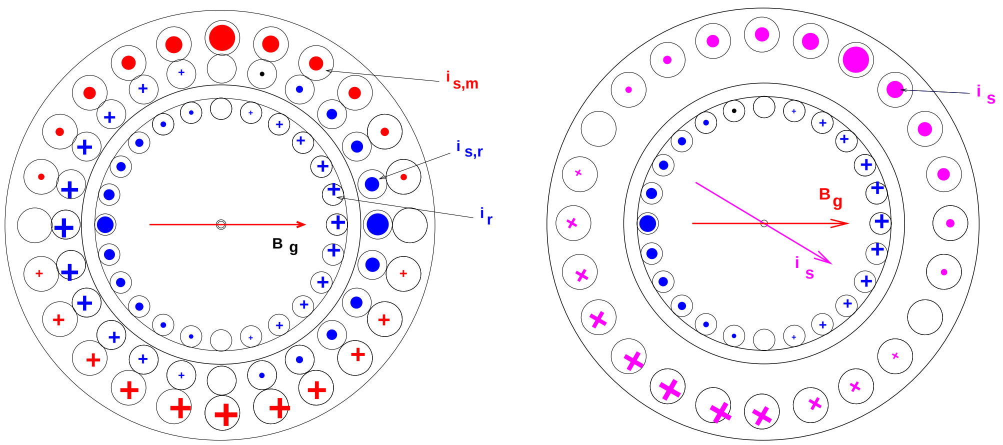


Fig. 6.7 Stator Magnetizing Current, airgap flux and rotor currents

These rotor currents, $\mathbf{i_r}$ produce additional airgap flux, which is 90^0 out of phase of the magnetizing flux. But the stator voltage, $\mathbf{e_s}$, is applied externally and it is proportional to and 90^0 out of phase of the airgap flux. Additional currents, $\mathbf{i_{sr}}$ will flow in the stator windings in order to cancel the flux due to the rotor currents. These currents are shown in figures 6.8. In 6.9 the corresponding space vectors are shown.



(a) Rotor Current and Stator Current Components

(b) Space Vectors of the Rotor and Stator Currents and induced voltages

Fig. 6.8 Rotor and Stator Currents in an Induction Motor

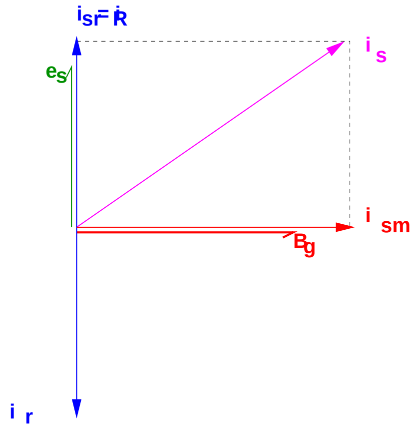


Fig. 6.9 Space-Vectors of the Stator and Rotor Current and Induced Voltages

There are a few things we should observe here:

- $\mathbf{i}_{sr}$ is 90^0 ahead of $\mathbf{i}_{sm}$, the stator magnetizing current. This means that it corresponds to currents in windings i_{1r}, i_{2r}, i_{3r}, leading by 90^0 the magnetizing currents i_{1m}, i_{2m}, i_{3m}.
- The amplitude of the magnetizing component of the stator current is proportional to the stator frequency, f_s and induced voltage. On the other hand, the amplitude of this component of the stator currents, $\mathbf{i}_{sr}$, is proportional to the current in the rotor, $\mathbf{i}_r$, which in turn is proportional to the flux and the slip speed, $\omega_r = \omega_s - \omega_0$, or proportional to the developed torque.
- We can, therefore split the stator current of one phase, i_{s1}, into two components: One in phase with the voltage, i_{sr1} and one 90^0 behind it, i_{sm1}. The first reflects the rotor current, while the second depends on the voltage and frequency. In an equivalent circuit, this means that i_{sr1} will flow through a resistor, and i_{sm1} will flow through an inductor.
- Since i_{sr1} is equal to the rotor current (through a factor), it will be inversely proportional to $\omega_s - \omega_r$, or, better, proportional to $\omega_s/(\omega_s - \omega_r)$. Figure 6.10 reflects these considerations.

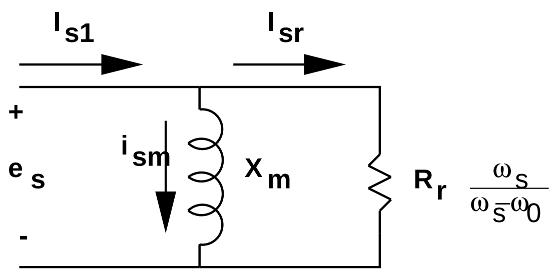


Fig. 6.10 Equivalent circuit of one stator phase

If we supply our induction motor with a three-phase, balanced sinusoidal voltage, we expect that the rotor will develop a torque according to equation 6.11. The relationship between speed, ω_0 and torque around synchronous speed is shown in figure 6.11. This curve is accurate as long as the speed does not vary more than $\pm 5\%$ around the rated synchronous speed ω_s.

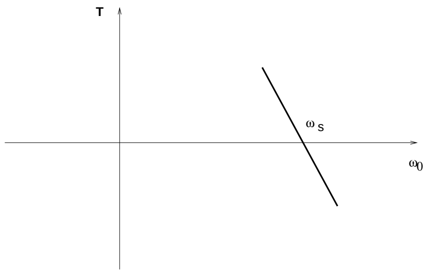


Fig. 6.11 Torque-speed characteristic near synchronous speed

We notice in 6.11 that when the speed exceeds synchronous, the torque produced by the machine is of opposite direction than the speed, i.e. the machine operates as a generator, developing a torque opposite to the rotation (counter torque) and transferring power from the shaft to the electrical system.

We already know the relationship of the magnetizing current, I_{sm} to the induced voltage E_{sm} through our analysis of the three-phase windings. Let us now relate the currents $\mathbf{i_r}$ and $\mathbf{i_{sr}}$ with the same induced voltage.

The current density on the rotor conducting sheet $\overrightarrow{J}$ is related to the value of the airgap flux density $\overrightarrow{B}_g$ through:

$$\overrightarrow{J} = \frac{1}{\rho}(\omega_s - \omega_0)\overrightarrow{B}_g \tag{6.12}$$

This current density corresponds to a space vector $\mathbf{i_r}$ that is opposite to the $\mathbf{i_{sr}}$ in the stator. This current space vector will correspond to the same current density:

$$J = i_{sr} N_s \frac{1}{r\,d} \tag{6.13}$$

while the stator voltage $\mathbf{e_s}$ is also related to the flux density B_g. Its amplitude is:

$$e_s = \omega_s \frac{\pi}{2} N_s l r B_g \tag{6.14}$$

Finally, substituting into 6.12, and relating phasors instead of space vectors, we obtain:

$$\mathbf{E}_s = R_R \frac{\omega_s}{\omega_s - \omega_o} \mathbf{I}_{sr} \tag{6.15}$$

Using this formulation we arrive at the formula for the torque:

$$T = 3\frac{E_s^2}{\omega_s}\frac{1}{R_R}\frac{\omega_s - \omega_0}{\omega_s} = 3\frac{\Lambda_s^2}{R_R}\omega_r = \frac{3P_g}{\omega_s} \tag{6.16}$$

where $\Lambda = (E_s/\omega_s))$ Here P_g is the power transferred to the resistance $R_R \frac{\omega_s}{\omega_s - \omega_0}$, through the airgap. Of this power a portion is converted to mechanical power represented by losses on resistance $R_R \frac{\omega_0}{\omega_s - \omega_0}$, and the remaining is losses in the rotor resistance, represented by the losses on resistance R_R. Figure 6.12 shows this split in the equivalent circuit. Note that the resistance $R_R \frac{\omega_0}{\omega_s - \omega_0}$ can be negative, indicating that mechanical power is absorbed in the induction machine.

6.4.1 Example

A 2-pole three-phase induction motor is connected in Y and is fed from a $60Hz$, $208V(l-l)$ system. Its equivalent one-phase rotor resistance is $R_R = 0.1125\Omega$. At what speed and slip is the developed torque $28Nm$?

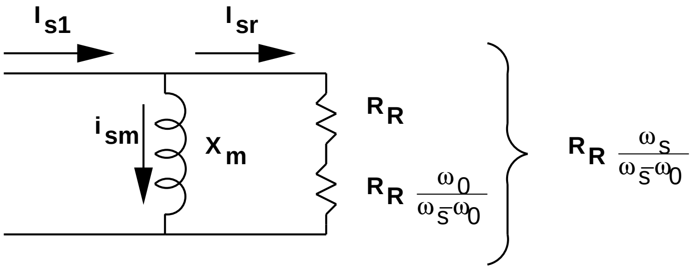


Fig. 6.12 Equivalent circuit of one stator phase separating the loss and torque rotor components

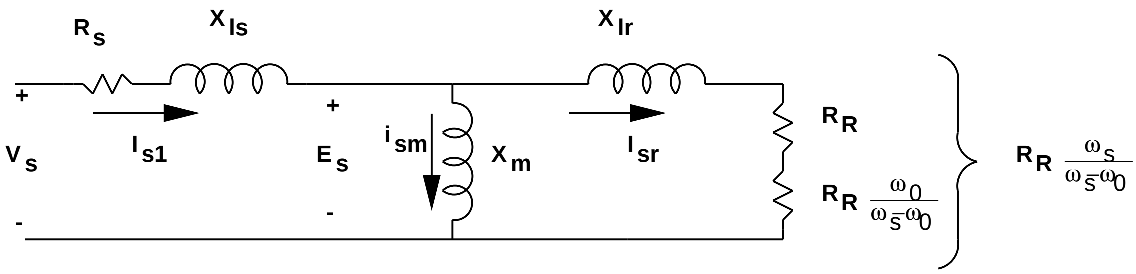


Fig. 6.13 Complete equivalent circuit of one stator phase

$$
\begin{aligned}
T &= 3\left(\frac{V_s}{\omega_s}\right)^2 \frac{1}{R_R}\omega_r \;\; \textit{with} \;\; V_s = 120V \\
28 &= 3\left(\frac{120}{377}\right)^2 \frac{1}{0.1125}\omega_r \Rightarrow \omega_r = 10.364\; rad/s \\
s &= \frac{\omega_r}{\omega_s} = \frac{10.364}{377} = 0.0275 \\
\omega_o &= \omega_s - \omega_r = 366.6 rad/s
\end{aligned}
$$

6.5 LEAKAGE INDUCTANCES AND THEIR EFFECTS

In the previous discussion we assumed that all the flux crosses the airgap and links both the stator and rotor windings. In addition to this flux there are flux components which link only the stator or the rotor windings and are proportional to the currents there, producing voltages in these windings 90^0 ahead of the stator and rotor currents and proportional to the amplitude of these currents and their frequency.

This is simple to model for the stator windings, since the equivalent circuit we are using is of the stator, and we can model the effects of this flux with only an inductance. The rotor leakage flux can be modelled in the rotor circuit with an inductance L_{ls}, as well, but corresponding to frequency of $f_r = \frac{\omega_s - \omega_0}{2\pi}$, the frequency of the rotor currents. It turns out that when we try to see its effects on the stator we can model it with an inductance L_{lr} at frequency f_s, as shown in the complete 1-phase equivalent circuit in figure 6.13.

Here $\mathbf{E_s}$ is the phasor of the voltage induced into the rotor windings from the airgap flux, while $\mathbf{V}_s$ is the phasor of the applied 1-phase stator voltage. The torque equations discussed earlier, 6.16, still hold, but give us slightly different results: We can develop torque-speed curves, by selecting speeds, solving the equivalent circuit, calculating power P_g, and using equation 6.16 for the torque. Figure 6.14 shows these characteristics for a wide range of speeds.

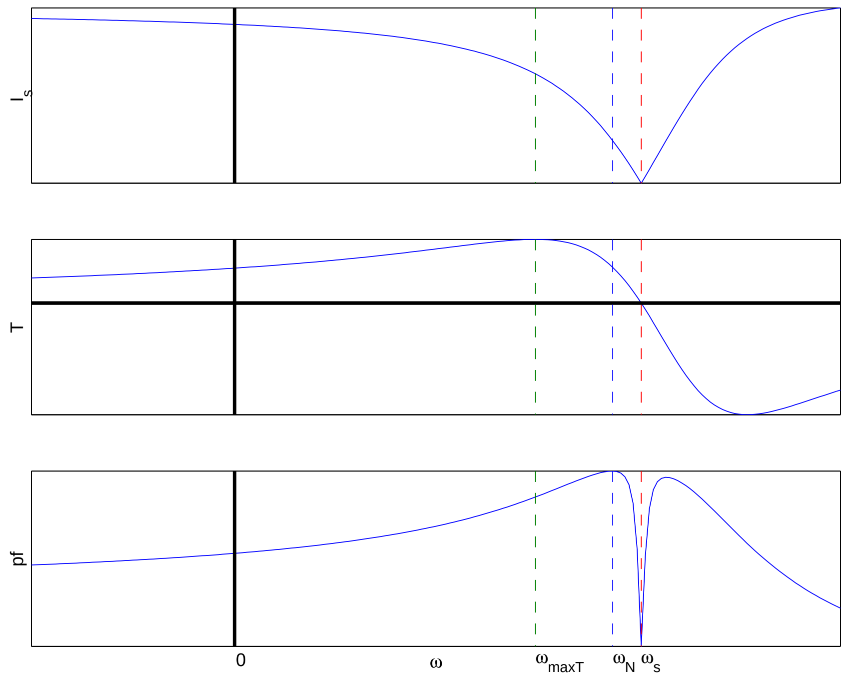


Fig. 6.14 Torque, current and power factor of an induction motor vs. speed

6.6 OPERATING CHARACTERISTICS

Figure 6.14 shows the developed torque, current, and power factor of an induction motor over a speed range from below zero (slip > 1) to above synchronous (slip < 0). It is clear that there are three areas of interest:

1. For speed $0 \leq \omega_o \leq \omega_s$ the torque is of the same sign as the speed, and the machine operates as a motor. There are a few interesting point on this curve, and on the corresponding current and p.f. curves.

2. For speed $\omega_0 \leq 0$, torque and speed have opposite signs, and the machine is in breaking mode. Notice that the current is very high, resulting in high winding losses.

3. for speed $\omega_o \geq \omega_s$ the speed and torque are of opposite signs, the machine is in generating mode, and the current amplitude is reasonable.

Let us concentrate now on the region $0 \leq \omega_o \leq \omega_s$. The machine is often designed to operate as a motor, and the operating point is near or exactly where the power factor is maximized. It is for this

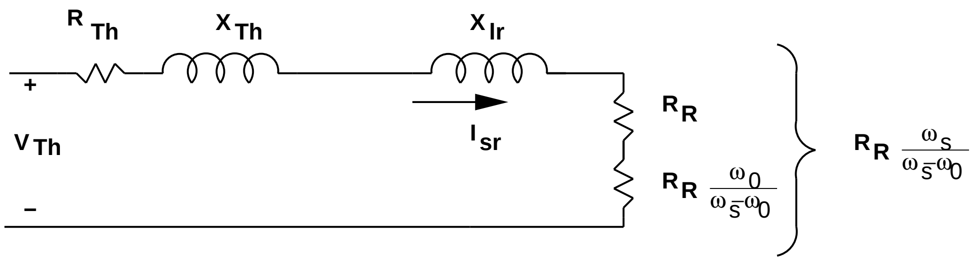


Fig. 6.15 Equivalent circuit of the stator with Thevenin equivalent of the stator components

point that the motor characteristics are given on the nameplate, rated speed, current, power factor and torque. When designing an application it is this point that we have to consider primarily: Will the torque suffice, will the efficiency and power factor be acceptable?

A second point of interest is starting, (slip $s = 1$) where the torque is not necessarily high, but the current often is. When selecting a motor for an application, we have to make sure that this starting torque is adequate to overcome the load torque, which may also include a static component. In addition, the starting current is often 3-5 times the rated current of the machine. If the developed torque at starting is not adequately higher than the load starting torque, their difference, the accelerating torque will be small and it may take too long to reach the operating point. This means that the current will remain high for a long time, and fuses or circuit breakers may operate.

A third point of interest is the maximum torque, T_{max}, corresponding to speed ω_{Tmax}. We can find it by analytically calculating torque as a function of slip, and equating the derivative to 0. This point is interesting, since points to the right of it correspond in general to stable operating conditions, while point to its left correspond to unstable operating conditions.

We can study this point if we take the Thevenin equivalent circuit of the left part of the stator equivalent circuit, including, V_s, R_s, X_{ls} and X_m. This will give us the circuit in figure 6.15.

Using the formula 6.16 we arrive at:

$$T = 3\frac{1}{\omega_s}\frac{V_{Th}^2\left(\frac{R_R}{s}\right)}{\left(R_{Th}+\frac{R_R}{s}\right)^2+\left(X_{Th}+X_{lr}\right)^2} \tag{6.17}$$

The maximum torque will develop when the airgap power, P_g, i.e. the power delivered to R_R/s, is maximum, since the torque is proportional to it. Taking derivative of 6.17, we find that maximum torque will occur when:

$$\frac{R_R}{s_{maxT}} = \sqrt{\left(R_{Th}\right)^2+\left(X_{Th}+X_{lr}\right)^2} \quad \text{or} \tag{6.18}$$

$$s_{maxT} = \frac{R_R}{\sqrt{R_{Th}^2+\left(X_{Th}+X_{lr}\right)^2}} \tag{6.19}$$

giving maximum torque:

$$T_{max} = 3\frac{1}{2\omega_s}\frac{V_{Th}^2}{R_{Th}+\sqrt{R_{Th}^2+\left(X_{Th}+X_{lr}\right)^2}} \tag{6.20}$$

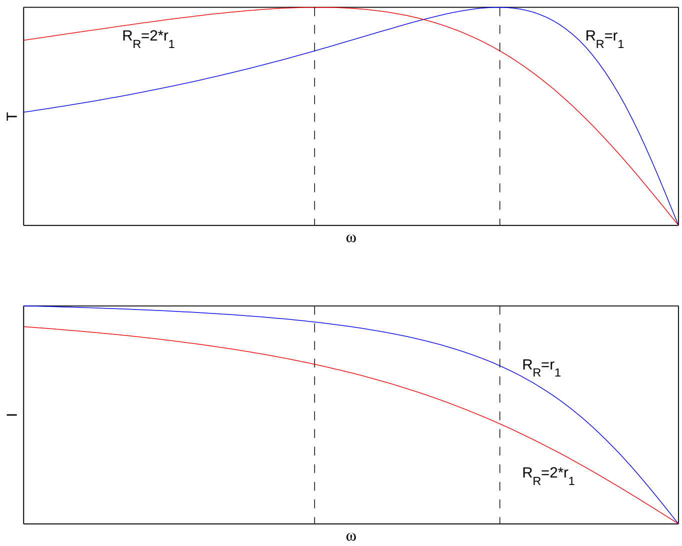


Fig. 6.16 Effect of changing rotor resistance on the torque-speed and current speed characteristic

If we neglect the stator resistance we can easily show that the general formula for the torque becomes:

$$T = T_{max} \frac{2}{\frac{s}{s_{Tmax}} + \frac{s_{Tmax}}{s}} \tag{6.21}$$

If we neglect both the stator resistance and the magnetizing inductance, we can develop simple formulae for T_{max} and ω_{Tmax}. To do so we have to assume operation near synchronous speed, where that value of $R_R \frac{\omega_s}{\omega_s - \omega_o}$ is much larger than $\omega_s L_{lr}$.

$$\omega_{Tmax} = \omega_s - \frac{R_R}{L_{lr} + L_{ls}} \tag{6.22}$$

$$T_{max} \simeq \frac{3}{2}\left(\frac{V_s}{\omega_s}\right)^2 \frac{1}{R_R}(\omega_s - \omega_{Tmax}) = \frac{3}{2}\left(\frac{V_s}{\omega_s}\right)^2 \frac{1}{L_{ls} + L_{lr}} \tag{6.23}$$

We notice here that the slip frequency at this torque, $\omega_r = \omega_s - \omega_{Tmax}$, for a constant flux $\Lambda_s = \frac{E_s}{\omega_s}$ is independent of frequency and proportional to the resistance R_R. We already know that this resistance is proportional to the rotor resistance, so if the rotor resistance is increased, the torque-speed characteristic is shifted to the left, as shown in figure 6.16.

If we have convenient ways to increase the rotor resistance, we can increase the starting torque, while decreasing the starting current. Increasing the rotor resistance can be easily accomplished in a wound-rotor machine, and more complex in squirrel cage motor, by using double or deep rotor bars.

In the formulae developed we notice that the maximum torque is a function of the flux. This means that we can change the frequency of the stator voltage, but as long as the voltage amplitude changes so that the flux stays the same, the maximum torque will also stay the same. Figure 6.17 shows this. This is called *Constant Volts per Hertz Operation* and it is a first approach to controlling the speed of the motor through its supply.

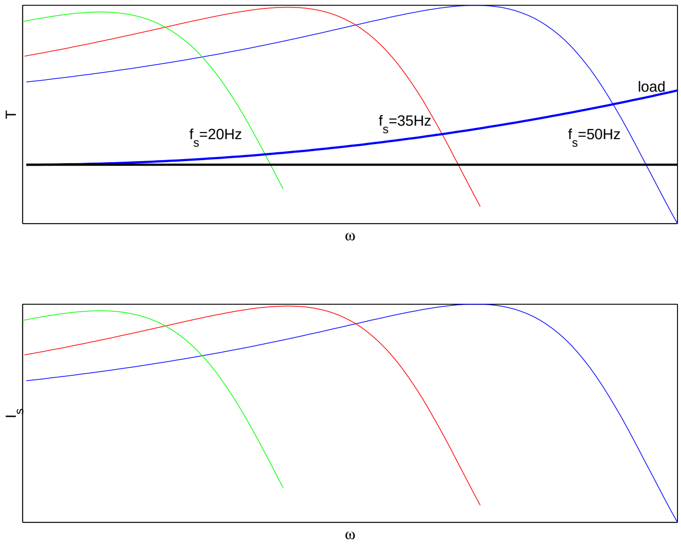


Fig. 6.17 Effect on the Torque - Speed characteristic of changing frequency while keeping flux constant

Near synchronous speed the effect of the rotor leakage inductance can be neglected, as discussed earlier. This assumption gives us the approximate torque-speed equation 6.16 discussed earlier.

$$T = 3\frac{E_s^2}{\omega_s}\frac{1}{R_R}\frac{\omega_s - \omega_0}{\omega_s} = 3\frac{\Lambda_s^2\omega_r}{R_R} = \frac{3P_g}{\omega_s}$$

Figure 6.18 shows both exact and approximate torque-speed characteristics. It is important to notice that the torque calculated from the approximate equation is grossly incorrect away from synchronous speed.

6.7 STARTING OF INDUCTION MOTORS

To avoid the problems associated with starting (too high current, too low torque), a variety of techniques are available.

An easy way to decrease the starting current is to decrease the stator terminal voltage. One can notice that while the stator current is proportional to the in voltage, torque will be proportional to its square. If a transformer is used to accomplish this, both developed torque and *line* current will decrease by the square of the turns ratio of the transformer.

A commonly used method is to use a motor designed to operate with the stator windings connected in Δ, and have it connected in Y at starting. As the voltage ratio is.

$$V_{s,Y} = \frac{1}{\sqrt{3}}V_{s,\Delta} \tag{6.24}$$

then

$$I_{s,Y} = \frac{1}{\sqrt{3}}I_{s,\Delta} \tag{6.25}$$

$$T_{s,Y} = \frac{1}{3}T_{s,\Delta} \tag{6.26}$$

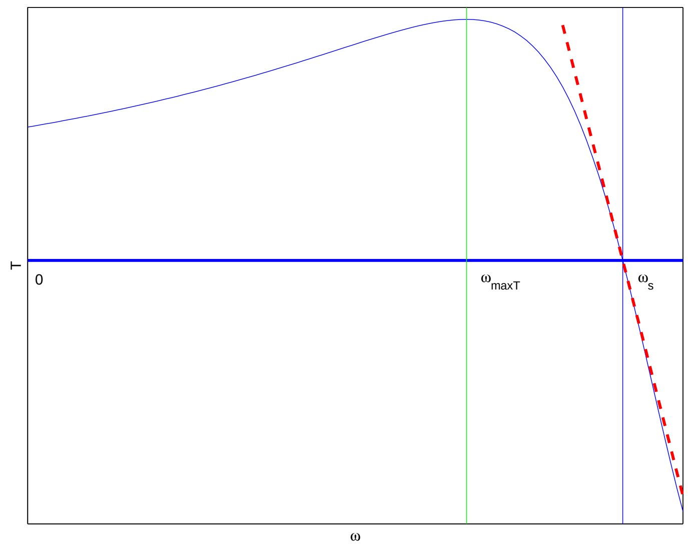


Fig. 6.18 Exact and approximate torque-speed characteristics

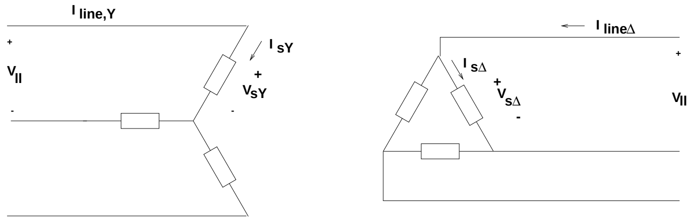


Fig. 6.19 Y- $Delta$ starting of an induction motor

But in a Δ connection, $I_{line} = \sqrt{3} I_{ph}$, leading to:

$$I_{line,Y} = \frac{1}{3} I_{line,\Delta} \tag{6.27}$$

Once the machine has approached the desired operating point, we can reconfigure the connection to Δ, and provide better efficiency.

This decrease in current is often adequate to allow a motor to start at low load starting torque. Using a variable frequency and voltage supply we can comfortably *increase* the starting torque, as shown in figure 6.17, while decreasing the starting current.

6.8 MULTIPLE POLE PAIRS

If we consider that an induction machine will operate close to synchronous speed (3000rpm for $50Hz$ and 3600rpm for $60Hz$) we may find that the speed of the machine is too high for an application. If we recall the pictures of the flux in AC machines we have seen, we can notice that the flux has a relatively long path to travel in the stator making the stator heavy and lossy.

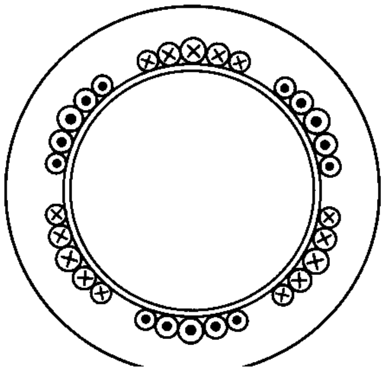


Fig. 6.20 Equivalent windings for a 6 pole induction motor

A machine with more than one pole pair is quite similar to that with only one. The difference is that for example in a 4 pole machine each side of a sinusoidally distributed winding of one phase covers only 90^0 instead of 180^0. A result is that there is room for four rather than two coil sides of each phase. Figure 6.20 shows at one instant the equivalent windings resulting from the the three phase windings.

The effects of a large number of poles on the operation of the machine are easy to predict. If the machine has p poles, or $p/2$ pairs of poles, in one period of the voltage the flux will travel $\frac{2}{p}w_s \; rad/s$. Hence the rotor speed corresponding to synchronous will be ω_{sm}:

$$\omega_{sm} = \frac{2}{p}\omega_s \tag{6.28}$$

We introduce now the actual, mechanical speed of the rotor, ω_m, while we keep the term ω_o as the rotor speed of a two pole motor. We generally measure ω_m in rad/s, while we measure ω_0 in *electrical* rad/s. We retain the same definition for slip based on the electrical speed ω_0 .

$$\omega_m = \frac{2}{p}\omega_o \tag{6.29}$$

$$s = \frac{\omega_s - \omega_0}{\omega_s} = \frac{\omega_s - \frac{p}{2}\omega_m}{\omega_s} \tag{6.30}$$

This means that for a 4 pole machine, supplied from a source of at $60Hz$, and operating close to rated conditions, the speed will be near $1800rpm$, while for a 6 pole machine, the speed will be near $1200rpm$.

While increasing the number of poles results in a decrease of the synchronous and operating speeds of the machine, it also results in an increase of the developed torque of the machine by the same ratio. Hence, the corrected torque formula will be

$$T = 3\frac{p}{2}\frac{P_g}{\omega_s} = 3\frac{p}{2}\frac{P_m}{\omega_0} \tag{6.31}$$

Similarly, the torque near the synchronous speed is:

$$T = 3\frac{p}{2}\frac{E_s^2}{\omega_s}\frac{1}{R_R}\frac{\omega_s - \omega_0}{\omega_s} = 3\frac{p}{2}\frac{\Lambda_s^2\omega_r}{R_R} = 3\frac{p}{2}\frac{P_g}{\omega_s}$$

while the previously developed formulas for maximum torque will become:

$$T_{max} = 3\frac{p}{2}\frac{1}{2\omega_s}\frac{V_{Th}^2}{R_{Th} + \sqrt{R_{Th}^2 + (X_{Th} + X_{lr})^2}} \tag{6.32}$$

and

$$T_{max} \simeq \frac{3}{2}\frac{p}{2}\left(\frac{V_s}{\omega_s}\right)^2 \frac{1}{R_R}(\omega_s - \omega_{Tmax}) = \frac{3}{2}\frac{p}{2}\left(\frac{V_s}{\omega_s}\right)^2 \frac{1}{L_{ls} + L_{lr}} \tag{6.33}$$

6.8.1 Example

A 3-phase 2-pole induction motor is rated $190V$, $60Hz$, it is connected in Y, and has $R_r = 6.6\Omega$, $R_s = 3.1\Omega$, $X_M = 190\Omega$, $X_{lr} = 10\Omega$, and $X_{ls} = 3\Omega$. Calculate the motor starting torque, starting current and starting power factor under rated voltage. What will be the current and power factor if no load is connected to the shaft?

1. *At starting $s = 1$:*

$$\begin{aligned}
\mathbf{I}_s &= \frac{190}{\sqrt{3}} / \left\{[3.1 + j3] + j190 || (6.6 + j10)\right\} = 7.06^{\angle -54.5^0} A \\
\mathbf{I}_R &= \mathbf{I}_s \frac{j190}{6.6 + j10 + j190} = 6.7^{\angle -52.6^0} A \\
T &= 3\frac{P_{gap}}{\omega_s}\frac{p}{2} = 3\frac{6.7^2 \cdot 6.6}{377}\frac{2}{2} = 2.36Nm
\end{aligned}$$

2. *Under no load the speed is synchronous and $s = 0$:*

$$\begin{aligned}
\mathbf{I}_s &= 110 / [3.1 + j3 + j190] = 0.57^{\angle -89.1^0} A \\
I_s &= 0.57A \\
pf &= 0.016 \textit{lagging}
\end{aligned}$$

6.8.2 Example

A 3-phase 2-pole induction motor is rated $190V$, $60Hz$ it is connected in Y, and has $R_r = 6.6\Omega$, $R_s = 3.1\Omega$, $X_M = 190\Omega$, $X_{lr} = 10\Omega$, and $X_{ls} = 3\Omega$. It is operating from a variable speed - variable frequency source at a speed of 1910rpm, under a constant V/f policy and the developed torque is $0.8Nm$. What is the voltage and frequency of the source? (Hint: Calculate first the slip).

The ratio V_s/ω_s stays $110/377$.

$$\begin{aligned}
T &= \frac{p}{2}3\left(\frac{V_s}{\omega_s}\right)^2 \frac{1}{R_R}\omega_r \\
0.8 &= 1 \cdot 3\left(\frac{110}{377}\right)^2 \frac{1}{6.6}\omega_r \Rightarrow \omega_r = 20.65 \quad \textit{rad/s} \\
\omega_s &= \omega_m \frac{p}{2} + \omega_R = 220.66 \quad \frac{rad}{s} \\
\Rightarrow f_s &= 35Hz \Rightarrow V_s = 220.66\frac{110}{377} = 64.4V \quad \textit{or} \quad 110V_{l-l}
\end{aligned}$$

6.8.3 Example

A 3-phase 4-pole induction machine is rated $230V$, $60Hz$. It is connected in Y and it has $R_r = 0.191\Omega$, $R_s = 0.2\Omega$, $L_M = 35mH$, $L_{lr} = 1.5mH$, and $L_{ls} = 1.2mH$. It is operated as a generator *connected to a variable frequency/variable voltage source. Its speed is 2036rpm, with counter-torque of* $59Nm$*. What is the efficiency of this generator? (Hint: here power in is mechanical, power out is electrical; calculate first the slip)*

Although we do not know the voltage or the frequency, we know their ratio since it is always $132.8/377$.

$$\begin{aligned} T &= 3\frac{p}{2}\left(\frac{V_s}{\omega_s}\right)^2 \frac{1}{R_R}\omega_r \\ \Rightarrow -59 &= 3\cdot 2\left(\frac{132.8}{377}\right)^2 \frac{1}{0.191}\omega_r \\ \Rightarrow \omega_r &= -15.14 \quad \textit{rad/s} \end{aligned}$$

Now we can find the synchronous speed, by adding slip and rotor speeds:

$$\begin{aligned} \omega_s &= \omega_m\frac{p}{2} + \omega_r = \frac{2\pi\cdot 2036}{60}2 - 15.14 = 411.3 \quad \textit{rad/s} \\ \Rightarrow f_s &= 65.5Hz \Rightarrow V_s = 65.5\cdot\frac{132}{60} = 144V \end{aligned}$$

We have to recalculate the impedances of the equivalent circuit for the frequency of $65.5Hz$*:*

$$X_m = 35\cdot 10^{-3}\cdot 411.3 = 14.4\Omega, \quad X_{ls} = 0.49\Omega, \quad X_{lr} = 0.617\Omega$$

$$R_R\frac{\omega_r + \omega_m\frac{p}{2}}{\omega_r} = -5.38\Omega$$

$$\mathbf{I}_s = 144/\left[0.2 + j0.49 + j14.4||(0.191 - 5.38 + j0.617)\right] = 30^{\angle -148^0}A$$

$$\mathbf{I}_R = 27.2^{\angle -166.9}A$$

Notice that with generation operation $R_R < 0$*. We can calculate now losses etc.*

$$\begin{aligned} P_m &= 3\cdot 27.2^2 5.38 = 11.941kW \\ P_{rotor,loss} &= 3\cdot 27.2^2 0.191 = 423W \\ P_{stator,loss} &= 3\cdot 30^2 0.2 = 540W \\ \Rightarrow P_{out} &= P_m - P_{rotor,loss} - P_{stator,loss} = 10.980kW \\ \Rightarrow \eta &= \frac{P_{out}}{P_m} = 0.919 \end{aligned}$$

7

Synchronous Machines and Drives

We noticed in discussing induction machines that as the rotor approaches synchronous speed, the frequency of the currents in the rotor decreases, as does the amplitude of these currents. The reason an induction motor produces no torque at synchronous speed is not that the currents are DC, but rather that their amplitude is zero.

It is possible to operate a three-phase machine at synchronous speed if DC is externally applied to the rotor and the rotor is rotated at synchronous speed. In this case torque will be developed only at this speed, i.e. if the rotor is rotated at speeds other than synchronous, the average torque will be zero.

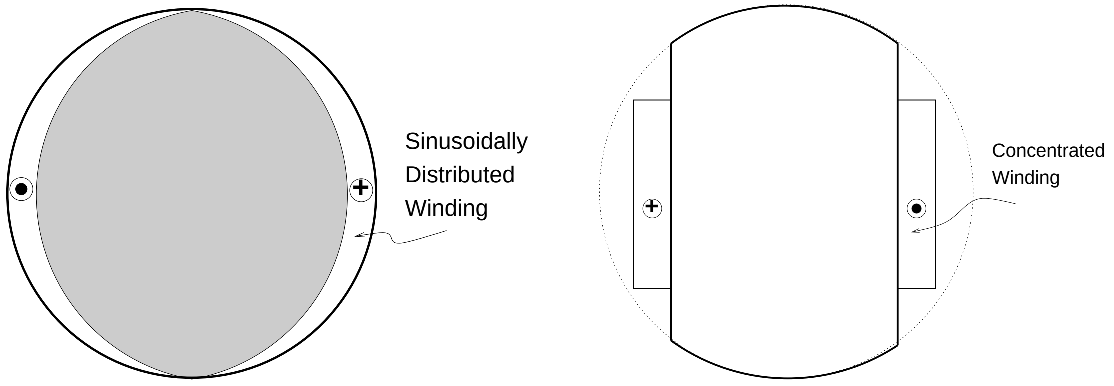
Machines operating on this principle are called synchronous machines, and cover a great variety. As generators they can be quite large, rated a few hundred MVA, and almost all power generation is through these machines. Large synchronous motors are not very common, but can be an attractive alternative to induction machines. Small synchronous motors with permanent magnets in the rotor, rather than coils with DC, are rapidly replacing induction motors in automotive, industrial and residential applications. since they are more efficient and lighter.

7.1 DESIGN AND PRINCIPLE OF OPERATION

The stator of a synchronous machine is of the type that we have already discussed, with three windings carrying a three-phase system of currents. The rotor can be one of two distinct types:

7.1.1 Wound Rotor Carrying DC

In this case the rotor steel structure can be either cylindrical, like that in figure 7.1a, or salient like the one in 7.1b. In either of these cases the rotor winding carries DC, provided to it through slip rings, or through a rectified voltage of an inside-out synchronous generator mounted on the same shaft. Here we'll limit ourselves to discussing only cylindrical rotors.



(a) Cylindrical rotor

(b) Salient pole rotor

Fig. 7.1

7.1.2 Permanent Magnet Rotor

In this case instead of supplying DC to the rotor we create a magnetic field attached to it by adding magnets on the rotor. There are many ways to do this, as shown in figure 7.2, and all have the following effects:

- The rotor flux can no longer be controlled externally. It is defined uniquely by the magnets and the geometry,
- The machine becomes simpler to construct, at least for small sizes.

7.2 EQUIVALENT CIRCUIT

The flux in the air gap can be considered to be due to two sources: the stator currents, and the rotor currents or permanent magnet. We have discussed already how the currents in the stator produce flux. Remember that this flux could also be produced by one equivalent winding, rotating at synchronous speed and carrying current equal to the magnitude of the stator-current space vector.

The rotor is itself such a winding, a real one, sinusoidally distributed, carrying DC and rotating at synchronous speed. It produces an airgap flux, which could also be produced by an additional set of three phase stator currents, giving a space vector $\mathbf{i_R}$. The amplitude of this space vector would be:

$$|\mathbf{i}_F| = \frac{N_s}{N_R} i_f \tag{7.1}$$

where N_s is the number of the stator turns of the one equivalent winding and N_R is the number of the turns in the rotor winding. Its angle ϕ_R would be the same as the angle of the rotor position:

$$\phi_R = \omega_s t + \phi_{R0} \tag{7.2}$$

The stator current space vector has amplitude:

$$|\mathbf{i}_s| = \frac{3}{2}\sqrt{2} I_s \tag{7.3}$$

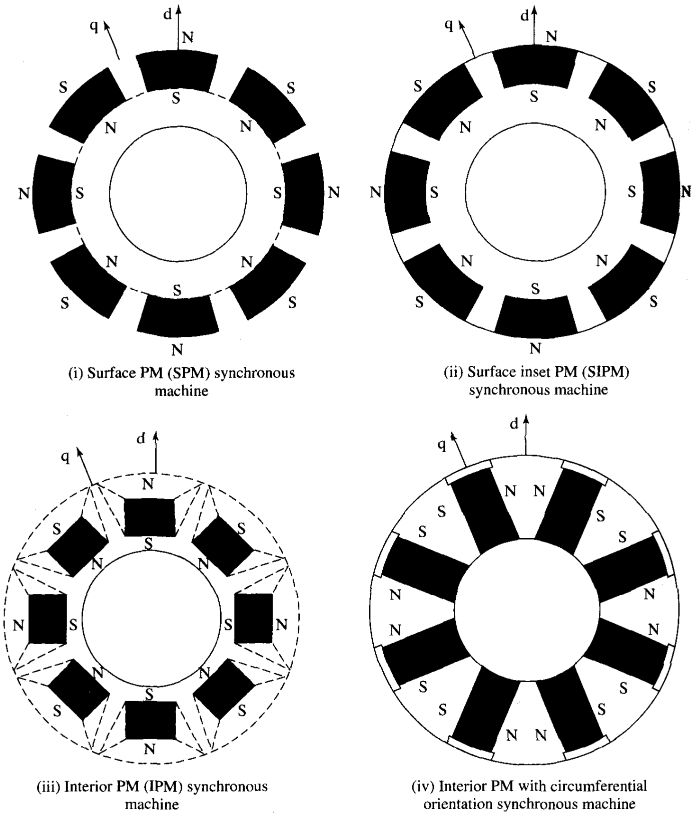


Fig. 7.2 Possible magnet placements in PMAC motors

where I_s is the rms current of one phase. The stator current space vector will have an instantaneous angle,

$$\phi_{is} = \omega_s t + \phi_{is0} \tag{7.4}$$

The airgap flux then is produced by both these current space vectors (rotor and stator). This flux induces in the stator windings a voltage, $\mathbf{e_s}$. In quasi steady-state everything is sinusoidal and the voltage space vector corresponds to three phase voltages $\mathbf{E_1}$, $\mathbf{E_2}$, $\mathbf{E_3}$. In this case we can create an equivalent circuit for the stator, 7.3. Here $\mathbf{I_F}$ is the stator AC current, that if it were to flow in the stator windings would have the same effects as the rotor current, i_f. In our analysis we can use as reference either the stator voltage, $\mathbf{V_s}$, or the stator current, $\mathbf{I_s}$. Figure 7.4. There are some angles to notice in this figure. We call θ the power factor angle, i.e. the angle between $\mathbf{I_s}$ and $\mathbf{V_s}$. We call β, the angle between $\mathbf{V_s}$ and $\mathbf{I_F}$, and power angle, δ, that between $\mathbf{I_F}$ and $\mathbf{I_S}$.

A few relationships to notice here:

- The space vector of the voltages induced in the stator, $\mathbf{e_s}$, is 90^0 ahead of the magnetizing current space vector, $\mathbf{i_M}$. This is so since $\mathbf{i_M}$ is what causes all the airgap flux that links the stator and induces $\mathbf{e_s}$. For a given frequency, the amplitude of this voltage, $\mathbf{e_s}$, is proportional to the current $\mathbf{i_M}$.

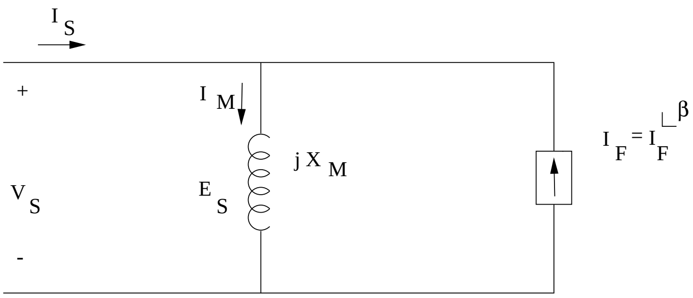


Fig. 7.3 Stator equivalent circuit for a synchronous machine

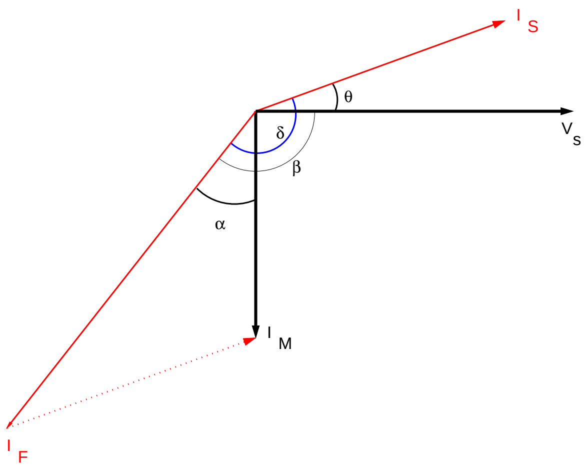


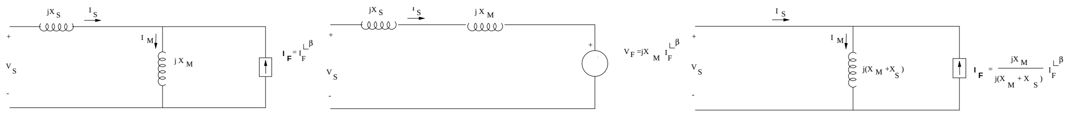
Fig. 7.4 Phasor diagram of an synchronous machine

- A permanent magnet machine can be considered equivalent to that with a winding, carrying a Direct Current, i_f, that is constant and cannot be controlled.

There are two modes of operation of a synchronous machine, that we'll study:

7.3 OPERATION OF THE MACHINE CONNECTED TO A BUS OF CONSTANT VOLTAGE AND FREQUENCY

This is usually the case for large synchronous generators or motors. We can consider any bus as one of constant voltage, by making a few modifications to the equivalent circuit as shown in figure 7.5.



(a) Original Equivalent Circuit with non-zero bus impedance

(b) The same circuit with Thevenin Equivalent of the Synchronous Machine

(c) Bqack to the Norton Equivalent circuit resulting in a modified Equivalent Circuit

Fig. 7.5 Accounting for system impedance in the model of a Synchronous Machine

Synchronous machines are very efficient, and most of the time we can neglect the stator resistance. All power then is converted to mechanical power and:

$$P = 3V_s I_s \cos\theta = T\omega_s \frac{2}{p} \tag{7.5}$$

$$P = -3V_s I_F \cos\beta \tag{7.6}$$

$$\mathbf{I_M} = \mathbf{I_s} + \mathbf{I_F} \tag{7.7}$$

$$\mathbf{V}_s = jX_m \mathbf{I_M} \tag{7.8}$$

In this operation V_s and ω_s (and therefore speed) remain constant. The only input variables are the torque, T, which affects output power, $P_{out} = T\omega_s \frac{2}{p}$, and the field current, i_f, which is proportional to I_F; the magnetizing current I_M is constant, since it is tied to the voltage V_s.

Let us assume that the machine is operated so that the power to it varies while the frequency and field current remain constant. Since this is a synchronous machine, the speed will not vary with the load. From equation 7.6 we can see that the power, and therefore the torque, varies sinusoidally with the angle β. Remember that β is the angle between the axis of the rotor winding, and the stator voltage space vector. Since this voltage space vector is 90^0 ahead of the space vector of the magnetizing current, $\beta - 90^0$ is the angle between the rotor axis and the magnetizing current space vector (same as the airgap flux). When there is no torque this angle is 0, i.e. the rotor rotates aligned with the flux, but when external torque is applied to the rotor in the direction of rotation the rotor will accelerate. As it accelerates (with the flux rotating at constant speed) the flux falls behind the rotor, and negative torque is developed, making the rotor slow down and rotate again at synchronous speed, but now ahead of the flux.

Similarly, when load torque is applied to the rotor, the rotor decelerates; as it does so, the angle β decreases beyond -90^0, i.e. the rotor falls behind the flux. Positive torque is developed that brings the rotor back to synchronous speed, but now rotating behind the stator flux.

In both cases when the load torque on a motor or the torque of the prime mover in a generator increases beyond a maximum, corresponding to $\cos\beta = \pm 1$, the machine cannot develop adequate torque and it loses synchronization.

Let us discuss now the effect of varying the field current while keeping the power constant. From equation 7.6, when power and voltage are kept constant, the product $I_F \cos\beta$ remains constant as well. But this product is the projection of $\mathbf{I_F}$ on the horizontal axis. This means that as the field current changes while power stays the same, the tip of $\mathbf{I_F}$ travels on a vertical line, as shown in figure 7.7a. Similarly, equation 7.5 means that at the same time the tip of $\mathbf{I_s}$ travels on another vertical line, also shown in figure 7.7a.

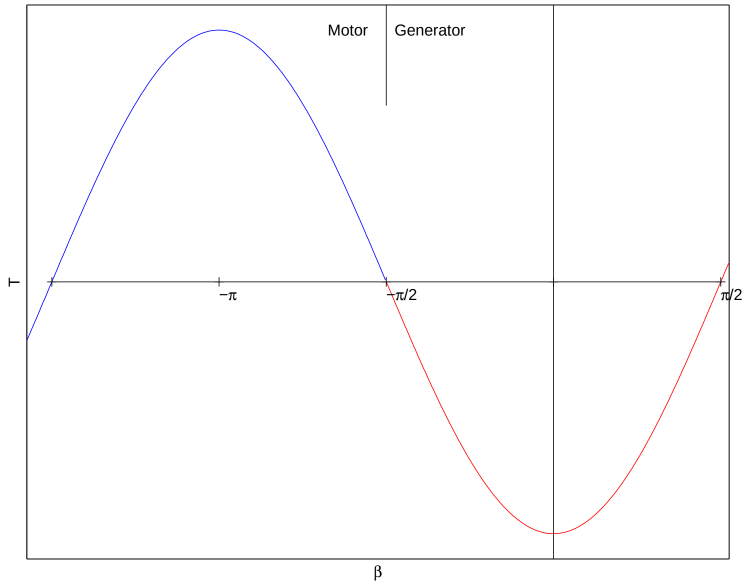
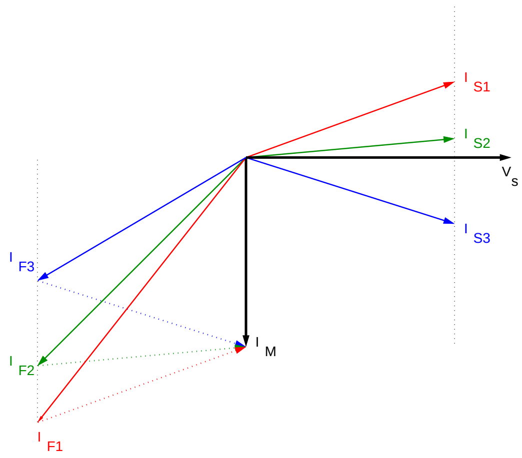
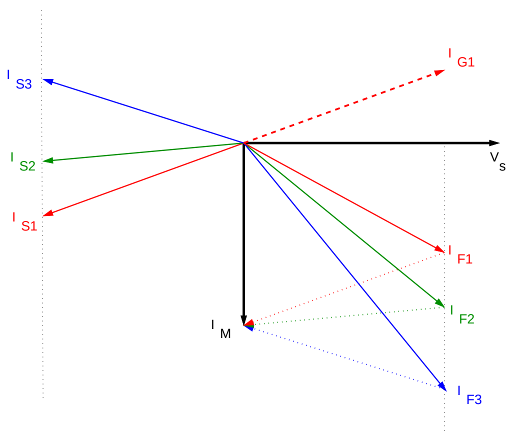


Fig. 7.6 Torque and angle β in a synchronous motor



(a) Varying the field current in a Synchronous motor under constant Power



(b) Varying the field current in a Synchronous Generator under constant Power

Fig. 7.7

It is clear from figure 7.7a that once the field current has exceeded a value specific to the power level, the power factor becomes leading and the machine produces reactive power. This is different from the operation of an induction machine, which always absorbs reactive power.

When the machine operates as a generator, the input power is negative. Figure 7.7b shows this operation for both leading and lagging load power factor. Here the angle between stator voltage and stator current defined in the direction shown in the equivalent circuit, is outside the range $-90^0 < \theta < 90^0$.

7.3.1 Example

A 3-phase Y-connected synchronous machine is fed from a $2300V$, $60Hz$. The ratio of the AC stator equivalent current to the rotor DC is $I_F/i_f = 1.8$ The magnetizing inductance of the machine is $200mH$.

- *The machine is operated as a motor and is absorbing $110kW$ at 0.89 p.f. leading. Calculate the required field current and the load angle. Draw the corresponding phasor diagrams.*

  Using figure 7.8:

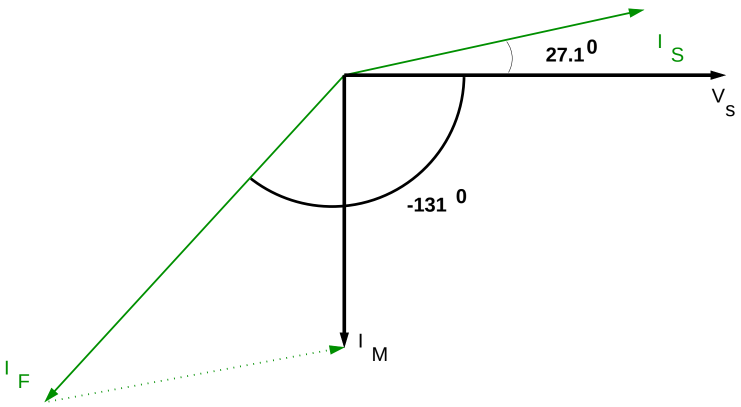


Fig. 7.8

$$X_M = 2\pi 60 \cdot 0.2 = 75.4\Omega$$
$$V_s = \frac{2300}{\sqrt{3}} = 1328V$$
$$\mathbf{I}_s = \frac{110 \cdot 10^3/3}{1328 \cdot 0.89}\angle 27.1^0 = 31^{\angle 27.1^0}A$$

from the stator voltage we can calculate $\mathbf{I}_M$ and from it $\mathbf{I}_F$.

$$\mathbf{I}_M = \frac{1328}{75.4} = 17.62^{\angle -90^0}A$$
$$\mathbf{I}_F = \mathbf{I}_M - \mathbf{I}_S = 42^{\angle -131^0}A$$
$$\Rightarrow i_f = \frac{I_F}{1.8} = 23.4A$$

- *Repeat for operation as generator at $110kW$, 0.82 pf leading.*

  Using figure 7.9:

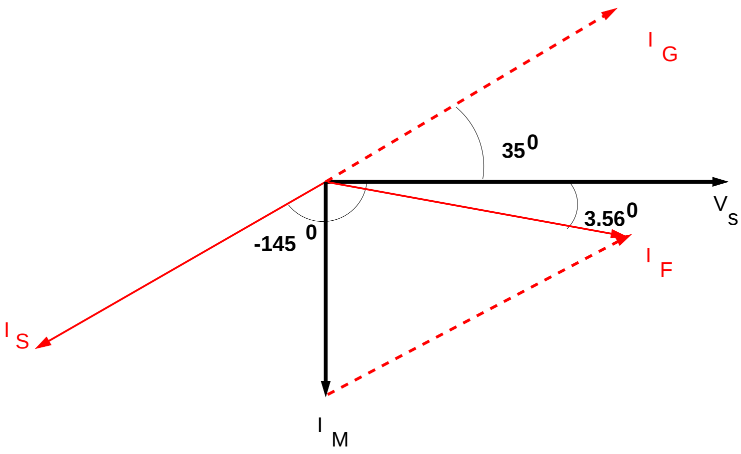


Fig. 7.9

$$\mathbf{I}_g = 33.7^{\angle 35} A \Rightarrow \mathbf{I}_S = 33.7^{\angle -145^0} A$$
$$\mathbf{I}_F = \mathbf{I}_M - \mathbf{I}_S = 27.66^{\angle 3.56^0} A$$
$$\Rightarrow i_f = 15.37A$$

- *What is the maximum power the machine above can produce (or absorb) when operating as a generator and at the field current just calculated?*

  We know that absorbed and produced power is:

  $$P = -3V_s I_F \cos\beta$$

  for $i_f = 15.37A$ *we have* $I_F = 27.66A$*, and* P *becomes maximum for* $\beta = 0$*, hence:*

  $$P = 3 \cdot 1328 \cdot 27.66 = 110.2kW$$

- *If the terminal voltage remains at* $2300V$*,* $60Hz$*, what is the minimal field current required to maintain operation as a motor with load* $70kW$*?*

  Again here:

  $$P = 3 \cdot V_s I_f \cos\beta = 3 \cdot 1328 \cdot I_F = 70 \cdot 10^3 W$$
  $$\Rightarrow I_F = 17.57A$$

7.4 OPERATION FROM A SOURCE OF VARIABLE FREQUENCY AND VOLTAGE

This operation requires that our synchronous machine is supplied by an inverter. The operation now is entirely different than before. We no longer have an infinite bus, but rather whatever stator voltage or current and frequency we desire. Moreover, with a space-vector controlled inverter, the phase of this voltage or current can be arbitrarily set at any instant i.e. we can define the stator current or voltage space vector, and obtain it at will. The considerations for the motor operation are also different:

- There is no concern for absorbing or supplying reactive power. Instead, there is a limit on the total stator current, determined by thermal considerations.
- There is a limit to the maximum voltage the source can supply, which leads to modifications of the machine mode of operation at high speeds.

Operation from source of variable frequency and voltage is most common for Permanent Magnet Machines, where the value of $|\mathbf{I_F}|$ is constant.

In simple terms, when the machine is starting as a motor the frequency applied should be zero, but the voltage space vector should be of such angle with respect to the rotor that torque is developed. as discussed in the previous section. As torque develops, the machine accelerates, and the applied stator currents have to create a rotating space vector leading the rotor flux. Voltage and frequency have to be increased, so that this torque is maintained. It is important therefore to monitor the position of the rotor in order to determine the location of the stator current or voltage space vector.

Two possible control techniques are implemented: either voltage control, where the stator voltage space vector is determined and applied, or current control, where he stator current space vector is applied.

For a fixed stator voltage and power (and torque) level, the stator losses are minimal when the stator voltage and current are in phase. Figure 7.10 shows this condition.

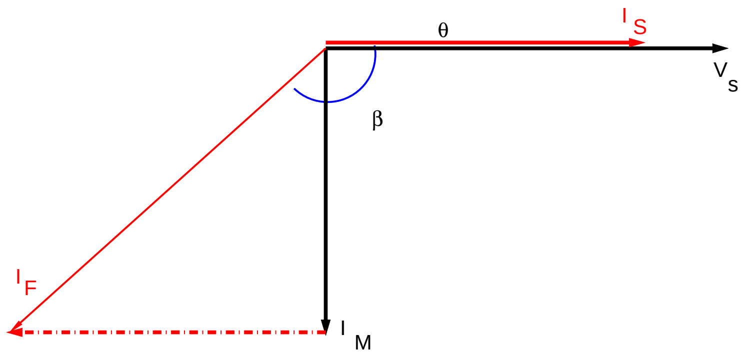


Fig. 7.10 Operation of a synchronous PM drive at constant voltage and Frequency

Notice that as the power changes with the voltage constant two things happen:

1. The voltage space vector varies in amplitude and the magnetizing current changes with it.
2. The amplitude of $\mathbf{I_F}$ stays constant, but its angle with respect to the voltage changes.

From the developed torque and speed we can calculate the frequency, the values of I_M and I_S, and the angle between the stator voltage space vector and the rotor, since

$$T = \frac{3p}{2\omega_s} V_s I_s = \frac{3p}{2} L_M I_M I_s \quad (7.9)$$

$$I_F^2 = I_M^2 + I_s^2 \quad (7.10)$$

and I_F is a constant in PM machines.

More common though is the case when the stator voltage is not constant. Here we monitor the position of the rotor and since the rotor flux and rotor space current are attached to it, we are actually monitoring the position of $\mathbf{I_F}$. To make matters simple we use this current rather than the stator voltage as reference, as shown in figure 7.11.

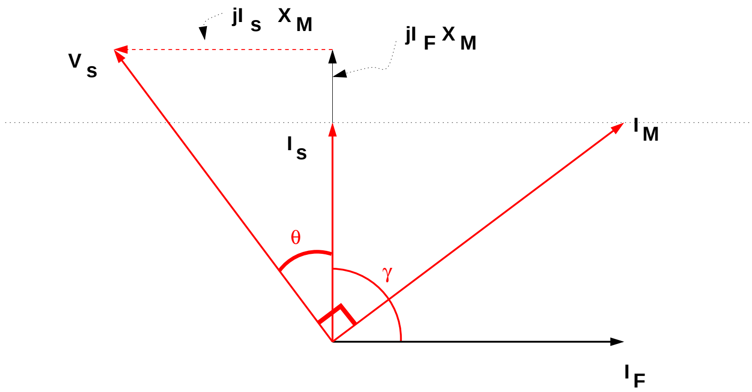


Fig. 7.11 Operation of a synchronous PM drive below base speed

Although previous formulae for power and torque are still true they are not as useful. We create new formulae that have the stator current I_s and magnetizing current I_M as variables. We also use the angle γ, between $\mathbf{I}_F$ and $\mathbf{I}_s$, since we can control it. Starting from what we already know:

$$P_g = \Re[\mathbf{V_s I_F^*}] = \Re[\mathbf{jX_M(I_F + I_s)I_F^*}] \tag{7.11}$$
$$= \Re[\mathbf{jX_M I_F I_F^*}] + \Re[\mathbf{jX_M I_s I_F^*}] \tag{7.12}$$
$$= X_M \Im[\mathbf{I_s I_F^*}] = \mathbf{X_M I_s I_F} \sin\gamma \tag{7.13}$$

For a given torque minimum losses require minimum value of the stator current. To minimize the value of I_s with constant power and I_F we choose $\gamma = 90^o$ and arrive at:

$$P_g = X_M \Im\left[\mathbf{I_s I_F^*}\right] = X_M I_s I_F \tag{7.14}$$
$$T = 3\frac{p}{2}\frac{P_g}{\omega_s} = 3\frac{p}{2} L_M \Im\left[\mathbf{I_s I_F^*}\right] = 3\frac{p}{2} L_M I_s I_F \tag{7.15}$$

which means that for constant power the projection of the stator current on an axis perpendicular to $\mathbf{I_M}$ is constant.

As the rotor speed increases, even if I_M stays constant, the stator voltage $V_s = \omega_s L_m I_s$ increases. At some speed ω_{sB}, the required voltage exceeds the maximum the power source can provide. We call this speed base speed; To increase the speed beyond it we no longer keep $\gamma = 90^o$. On the other hand at that speed we know that the voltage has reached its upper limit $V_s = V_{s,max}$, therefore the value of $I_M = V_{s,max}/X_M$ is known. In this case, equations 7.9 and 7.10 become:

$$T = \frac{3p}{2\omega_s} V_s I_s \cos\theta = \frac{3p}{2} L_M I_M I_s \cos\theta \tag{7.16}$$
$$I_F^2 = I_M^2 + I_s^2 + 2 I_M I_s \sin\theta \tag{7.17}$$

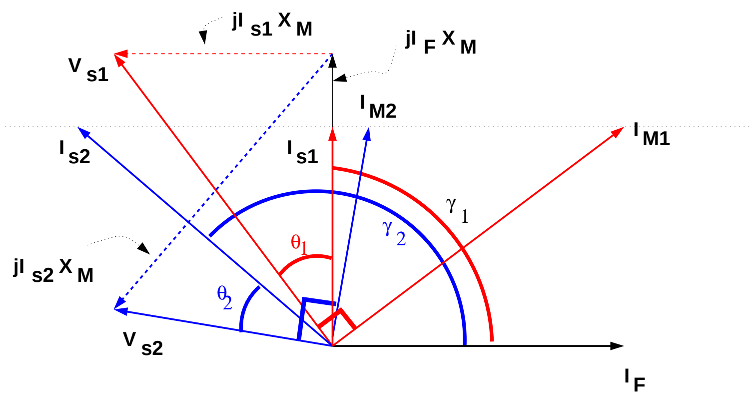


Fig. 7.12 Field weakening of a PM AC motor. The two diagrams at are at the same frequency, but the second one has $\gamma > 90^o$ and lower V_s

Figure 7.12 shows such an operation with the variables having the subscript 1. Note that we calculate torque from power:

$$P = 3X_M I_S I_F \sin\gamma \tag{7.18}$$

$$T = \frac{P}{\omega_s}\frac{p}{2} = 3\frac{p}{2}L_M \Im\left[\mathbf{I_S I_F^*}\right] = 3\frac{p}{2}L_M I_s I_F \sin\gamma \tag{7.19}$$

7.4.1 Example

A 3-phase, four pole, Y connected permanent magnet synchronous machine is rated $400V$, $50Hz$, $50kVA$. Its magnetizing inductance is $2.5mH$ and its equivalent field source current is $310A$. We can neglect stator resistance.

- *The machine is operated as a generator at rated frequency. Determine the maximum and minimum values of the stator phase voltage as the load current is varied from zero to rated value at unity power factor.*

  The rated phase voltage is $V_s = 400/\sqrt{3} = 231V$ and the rated stator current is $I_s = 50 \cdot 10^3/3 \cdot 231 = 72.2A$. With no load and at rated frequency the phase voltage is:

  $$V_s = \omega_s L_M I_F = 2\pi 50 \cdot 2.5 \cdot 10^{-3} \cdot 310 = 243.5V$$

  If the motor is operated at unity power factor, the stator current is collinear with the stator voltage, as in figure 7.13.

  From the current triangle:

  $$I_M^2 = I_F^2 - I_s^2 \Rightarrow I_M = \sqrt{310^2 - 72.7^2} = 301.5A$$

  and the stator voltage is:

  $$V_s = \omega_s L_M I_M = 236.8V$$

- *The machine is now operated as a variable speed drive motor from a variable voltage, variable frequency source. What should be the voltage and frequency in order to provide torque of $300Nm$ at 600rpm, if again we have unity power factor?*

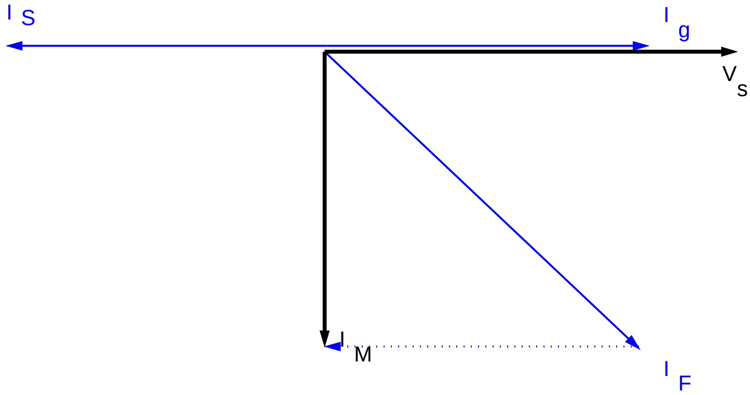


Fig. 7.13

The machine has four poles, so

$$\omega_s = \frac{p}{2} 600 \frac{2\pi}{60} = 20Hz$$

Torque can be expressed as a function of input power:

$$T = 3\frac{p}{2}\frac{1}{\omega_s} V_s I_s pf = 3\frac{p}{2\omega_s}(\omega_s L_M I_M) I_s = \frac{3p}{2} L_M I_M I_s = 300Nm$$

In addition to this equation we have from the current triangle for unity pf:

$$I_F^2 = I_M^2 + I_s^2 = 310^2$$

These two equations, solved together will give

$$\begin{aligned} I_M &= 303A \qquad I_s = 66A \qquad \textit{or} \\ I_M &= 66 \qquad I_s = 303A \end{aligned}$$

which leads to phase voltage and torque:

$$\begin{aligned} V_s &= \omega_s L_M I_M = 95.1V \\ P_m &= \frac{2}{p}\omega_s T = 18.84kW \end{aligned}$$

7.4.2 Example

A 2-pole, Y-connected, 3-phase Permanent Magnet synchronous generator is rated $230V$ (l-l) $10kVA$, $400Hz$. Its magnetizing inductance is $0.6mH$. First a test is performed: The rotor is externally driven at rated speed with the stator open circuited and the line-line voltage is measured at $240V$.

Based on the result of this test determine the stator voltage and power angle when the stator current, voltage and frequency are rated and the power factor of the load is 0.9 lagging.

From the test:

$$X_M = \omega_s L_M = 2\pi 400 \cdot 0.6 \cdot 10^{-3} = 1.508\Omega$$
$$V_s = |\mathbf{I_M X_M}| = \frac{240}{\sqrt{3}} \Rightarrow I_M = 91.9A$$

but at no load

$$I_F = I_M = 91.9A \qquad \text{and it is constant}$$

Now that we found I_F, to the problem: At the operating point

$$I_s = \frac{S}{\sqrt{3}V_{ll}} = \frac{10 \cdot 10^3}{\sqrt{3} \cdot 230} = 25.102A$$
$$pf = 0.9 \quad \Rightarrow \quad \theta = -25.84^o$$

from the geometry of the current triangle:

$$I_M^2 + I_s^2 - 2I_M I_S \cos\left(\frac{\pi}{2} + \theta\right) = I_F^2$$
$$\Rightarrow\Rightarrow I_M^2 + 25.1^2 + 2 \cdot 25.1 \cdot I_M \cdot 0.436 = 91.9^2$$
$$\Rightarrow I_M = 100A$$

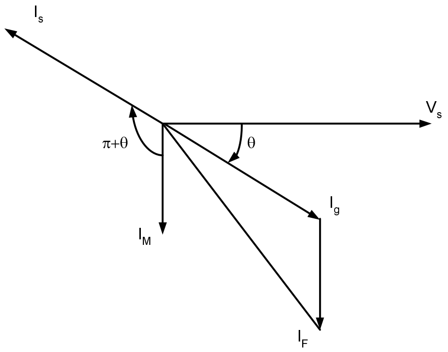


Fig. 7.14 Phasor diagram for this example

7.4.3 Example

A permanent magnet, Y connected, three-phase, 2-pole motor has $I_F = 40$ and $X_M = 0.9\Omega$ at $100Hz$.

1. *If it is absorbing $P = 1.5kW$ at $100Hz$ with minimum stator current I_s, calculate this current, the angle between $\mathbf{I_s}$ and $\mathbf{I_F}$, the speed, the stator voltage (line-neutral) and the power factor.*

   The minimum current I_s will exist when $\gamma = \angle(\mathbf{I_s}, \mathbf{I_F}) = 90^o$. Then:

$$P = 3X_M I_s I_F \quad \Rightarrow \quad I_s = \frac{1500}{3 \cdot 0.9 \cdot 40} = 13.89A$$
$$\Rightarrow \mathbf{I}_M = \mathbf{I}_F + \mathbf{I}_s = 40 + 13.89^{\angle 90^o} = 42.34^{\angle 19.15^o}A$$
$$\Rightarrow \mathbf{V}_s = j\omega_s L_M \mathbf{I}_M = 38.12^{\angle 109.15^o}V$$

the power factor is:

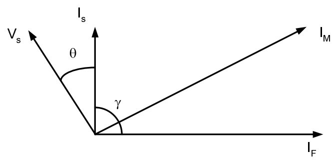
$$pf = \cos(109.14^o - 90^o) = 0.946 lagging$$

2. *It is desired to increase the motor speed to* $6900rpm$ *while keeping power the same,* $P = 1.5kW$*, but the supplied voltage has reached its upper limit of* $V_s = 38.12V$*. Now the motor absorbs the same power at at the voltage calculated in the previous question, but at frequency* $115Hz$*; This can be accomplished by having stator current no longer at a minimum value and* $\gamma \neq 90^o$*. Calculate again the angle between* $\mathbf{I_s}$ *and* $\mathbf{I_F}$*, the speed, and the power factor.*

$$P = -3V_sI_F\cos\beta \Rightarrow \cos\beta = -\frac{1500}{3\cdot 38.12\cdot 40} = -0.32$$
$$\Rightarrow \beta = -109.14^o \Rightarrow \mathbf{V_s} = 38.12^{\angle 109.14^o} A$$
$$\mathbf{I}_M = \frac{\mathbf{V}_s}{jX_M} = \frac{38.12^{\angle 109.15^o}}{j\frac{115}{100}0.9} = 36.83^{\angle 19.15^o} A$$
$$\mathbf{I_s} = \mathbf{I}_M - \mathbf{I}_F = 13.16^{\angle 113.18^o} A$$

the power factor is now

$$pf = \cos(109.14^o - 113.18^o) = 0.997 leading$$



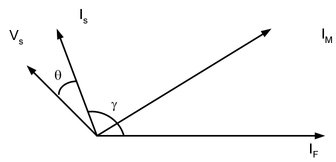


Fig. 7.15 Phasor diagram for this example

7.5 CONTROLLERS FOR PMAC MACHINES

Figure 7.16 shows a typical controller for an AC Machine. It requires a DC power supply, usually a rectifier fed from an AC source, an inverter and a controller.

Figure 7.17 shows in a slightly higher detail the controller

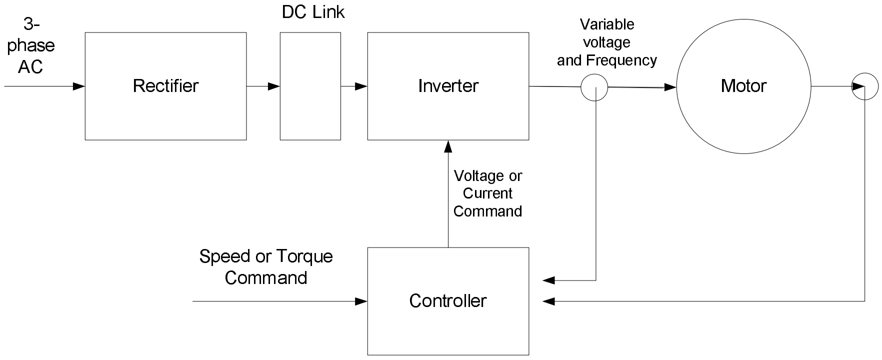


Fig. 7.16 Generic Controller for a PMAC Machine

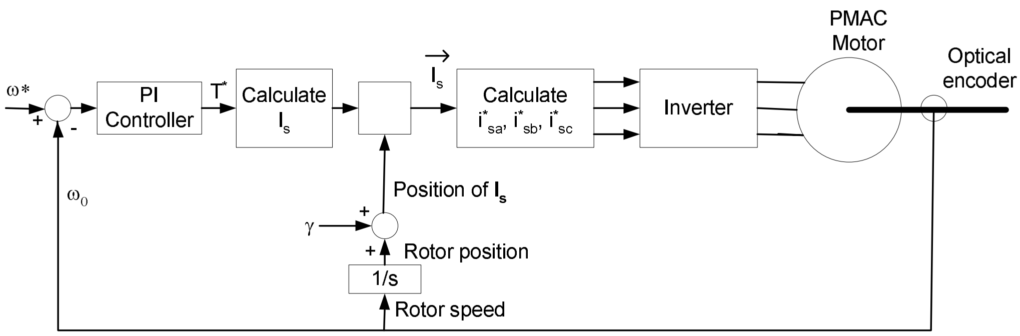


Fig. 7.17 Field Oriented controller for a PMAC Machine. The calculations for I_s are based on equation 7.19, and the calculation of $i^*_{sa}, i^*_{sb}, i^*_{sc}$ are calculated from the space vector $\mathbf{I_s}$ from equations 5.3

7.6 BRUSHLESS DC MACHINES

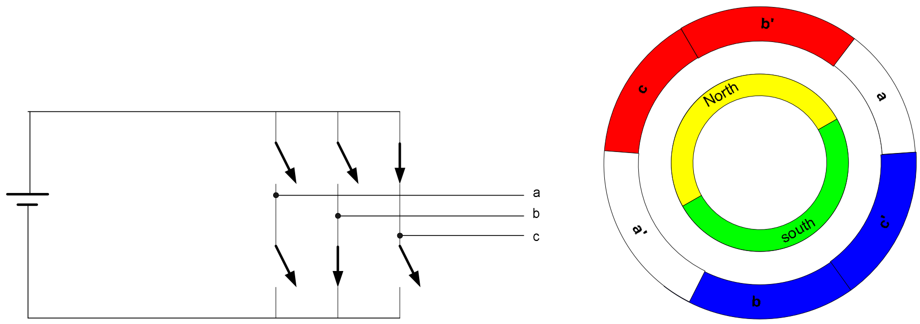
While it would be difficult to find the difference between a PM AC machine described above and a brushless DC machine by just looking at them, the concept of operation is quite different as is the analysis. The windings in the stator in a brushless DC machine are not sinusoidally distributed but instead they are concentrated, each occupying one third of the pole pitch. The flux density on the magnet surface and in the airgap is also not sinusoidally distributed over the magnet but almost uniform in the air gap.

As the stator currents interact with the flux coming from the magnet torque is developed. It should be clear that for the same direction of flux, currents in opposite directions result in opposite forces, and therefore in reduction of total torque. This in turn makes it necessary that all the current in the stator above the rotor is in the same direction. To accomplish this the following are needed:

- Sensors on the stator that sense the direction of the flux coming from the rotor,

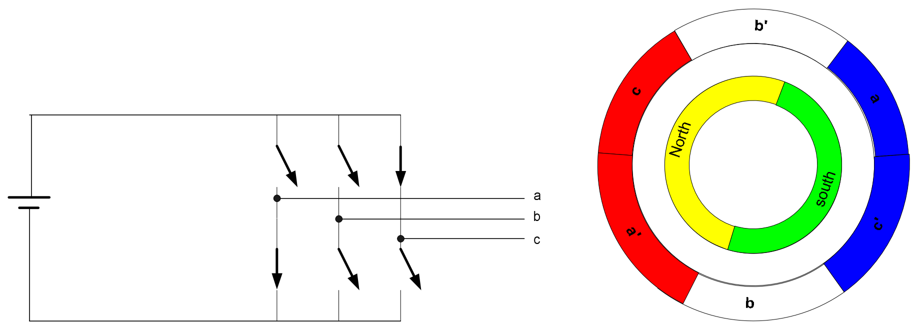
- A fast supply that will provide currents to the appropriate stator windings as determined by the flux direction.
- A way to control these currents, e.g. through Pulse Width Modulation
- A controller with inputs the desired speed, the flux direction in the stator and the stator currents, and outputs the desired currents in the stator

Figures 7.18 and 7.19 show the rotor positions, the stator currents and the switches of the supply inverter for two rotor positions.



(a) Switch positions (b) Machine cross section

Fig. 7.18 Energizing the windings in a brushless DC motor



(a) Switch positions (b) Machine cross section

Fig. 7.19 Energizing the windings in a brushless DC motor, a little later

The formulae that describe the operation of the system are quite simple. The developed torque is proportional to the stator currents:

$$T = k \cdot I_s \tag{7.20}$$

At the same time, the rotating flux induces a voltage in the energized windings:

$$E = k \cdot \omega \tag{7.21}$$

Finally the terminal voltage differs from the induced voltage by a resistive voltage drop:

$$V_{term} = E + I_s R \tag{7.22}$$

These equations are similar to those of a DC motor 4.4 - 4.6. This is the reason that although this machine is entirely different from a DC motor, it is called brushless DC motor.

8

Line Controlled Rectifiers

The idea here is to draw power from a 1-phase or 3-phase system to provide with DC a load. The characteristics of the systems here are among others, that the devices used will turn themselves off (commutate) and that the systems draw reactive power from the loads.

8.1 1- AND 3-PHASE CIRCUITS WITH DIODES

If the source is 1-phase, a diode is used and the load purely resistive, as shown in figure 8.1 things are simple. When the source voltage is positive, the current flows through the diode and the voltage of the source equals the voltage of the load. If the load includes an inductance and a source (e.g. a battery we want to charge), as in figure 8.2, the diode will continue to conduct even when the load voltage becomes negative as long as the current is maintained.

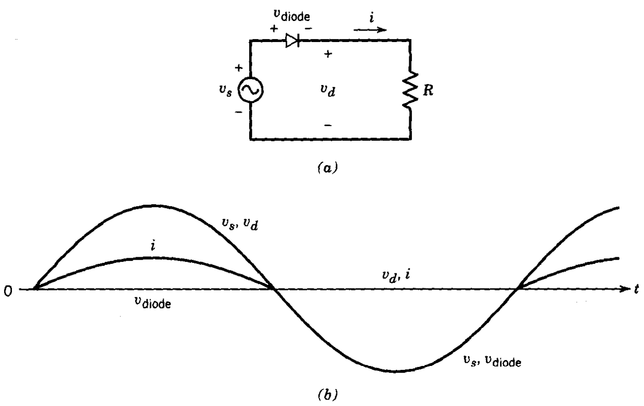


Fig. 8.1 Simple circuit with Diode and resistive Load

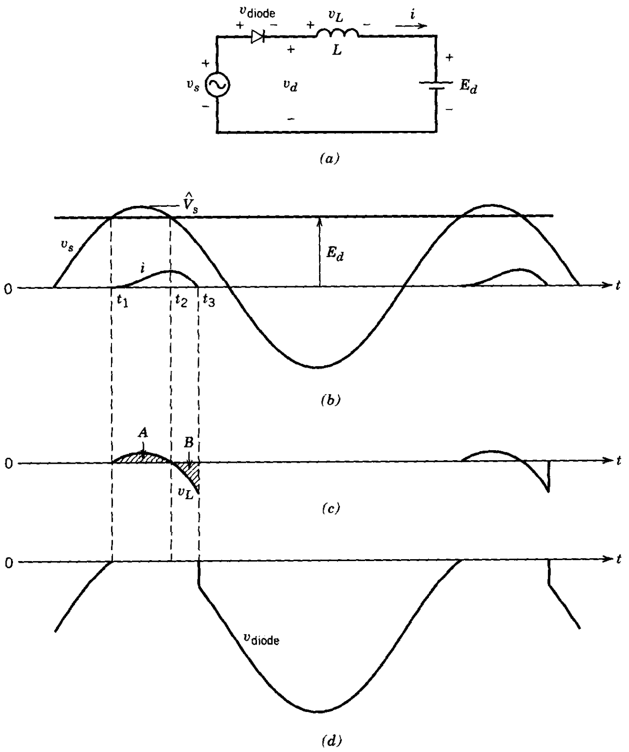


Fig. 8.2 Smple Circuit with Diode and inductive load with voltage source

8.2 ONE -PHASE FULL WAVE RECTIFIER

More common is a single phase diode bridge rectifier 8.3. The load can be modelled with one of two extremes: either as a constant current source, representing the case of a large inductance that keeps the current through it almost constant, or as a resistor, representing the case of minimum line inductance. We'll study the first case with AC and DC side current and voltage waveforms shown in figure 8.4.

If we analyze these waveforms, the output voltage will have a DC component V_{do}:

$$V_{do} = \frac{2}{\pi}\sqrt{2}V_s \simeq 0.9V_s \tag{8.1}$$

where V_s is the RMS value of the input AC voltage. On the other hand the RMS value of the output voltage will be

$$V_s = V_d \tag{8.2}$$

containing components of higher frequency.

Similarly, on the AC side the current is not sinusoidal, but rather it changes abruptly between I_d and $-I_d$.

$$I_{s1} = \frac{2}{\pi}\sqrt{2}I_d = 0.9I_d \tag{8.3}$$

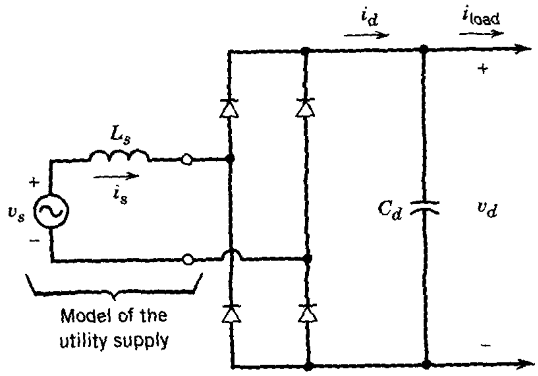


Fig. 8.3 One-phase full wave rectifier

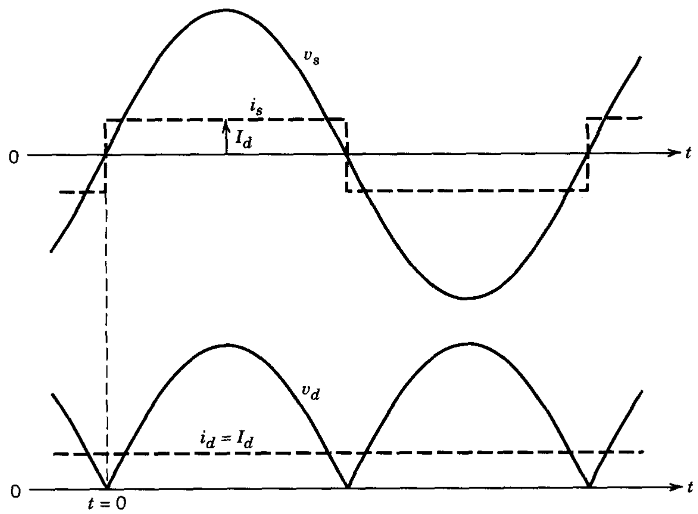


Fig. 8.4 Waveforms for a one-phase full wave rectifier with inductive load

and again the RMS values are the same

$$I_d = I_s \tag{8.4}$$

Giving a total harmonic distortion

$$THD = \frac{\sqrt{I_s^2 - I_{s1}^2}}{I_{s1}} \cong 48.43\% \tag{8.5}$$

It is important to notice that if the source has some inductance (and it usually does) commutation will be delayed after the voltage has reached zero, until the current has dropped to zero as shown in figure 8.5. This will lead to a decrease of the output DC voltage below what is expected from formula 8.1.

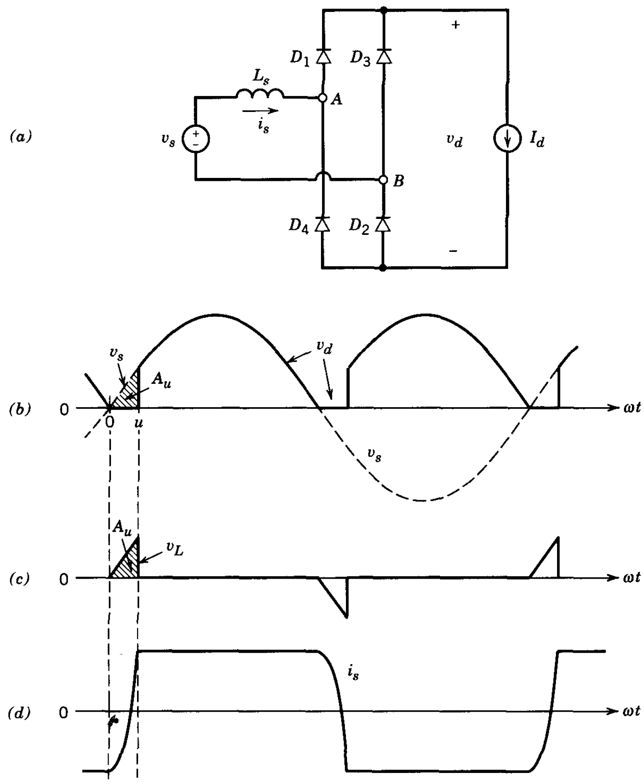


Fig. 8.5 One-phase full wave rectifier with inductive load and source inductance

8.3 THREE-PHASE DIODE RECTIFIERS

The circuit of figure 8.3 can be modified to handle three phases, without using 12 but rather 6 diodes, as shown in figure 8.6. Figure 8.7 shows the AC side currents and DC side voltage for the case of high load inductance. Similar analysis as before shows that on the DC side the voltage is:

$$V_{do} = \frac{3}{\pi}\sqrt{2}V_{LL} = 1.35V_{LL} \tag{8.6}$$

From figure 8.6 it is obvious that on the AC side the rms current, I_s is

$$I_s = \sqrt{\frac{2}{3}}I_d = 0.816I_d \tag{8.7}$$

while the fundamental current, i.e. the current at power frequency is:

$$I_{s1} = \frac{1}{\pi}\sqrt{6}I_d = 0.78I_d \tag{8.8}$$

Again, inductance on the AC side will delay commutation, causing a voltage loss, i.e. the DC voltage will be less than that predicted by equation 8.6.

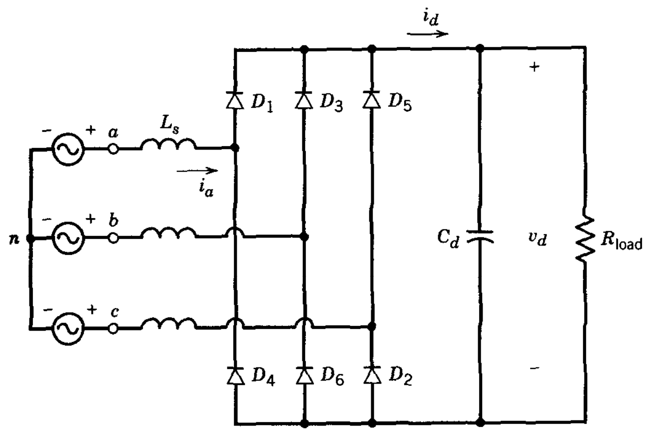


Fig. 8.6 Three-phase full-wave rectifier with diodes

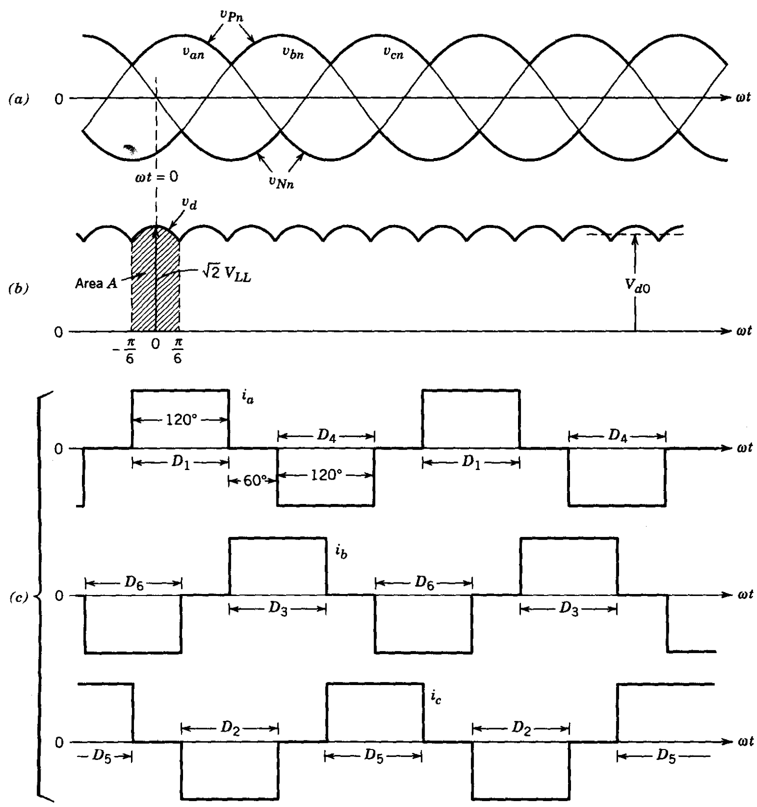


Fig. 8.7 Waveforms of a three-phase full-wave rectifier with diodes and inductive load

8.4 CONTROLLED RECTIFIERS WITH THYRISTORS

Thyristors give us the ability to vary the DC voltage. Remember that to make a thyristor start conducting, the thyristor has to be forward biased and a gate pulse provided to its gate. Also, to turn off a thyristor the current through it has to reverse direction for a short period of time, t_{rr}, and return to zero.

8.5 ONE PHASE CONTROLLED RECTIFIERS

Figure 8.8 shows the same 1-phase bridge we have already studied, now with thyristors instead of diodes, and figure 8.9 shows the output voltage and input current waveforms. In this figure α is the delay angle, corresponding to the time we delay triggering the thyristors after they became forward biased. Thyristors 1 and 2 are triggered together and of course so are 3 and 4. Each pair of thyristors is turned off immediately (or shortly) after the other pair is turned on by gating. Analysis similar to

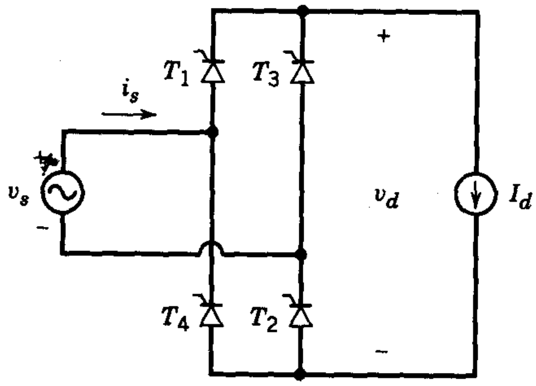


Fig. 8.8 One-phase full wave converter with Thyristors

that for diode circuits will give:

$$V_{do} = \frac{2}{\pi}\sqrt{2}V_s \cos\alpha = 0.9V_s \cos\alpha \tag{8.9}$$

and the relation for the currents is the same

$$I_{s1} = \frac{2}{\pi}\sqrt{2}I_d = 0.9I_d \tag{8.10}$$

We should notice in figure 8.9 that the current waveform on the AC side is offset i time with respect to the corresponding voltage by the same angle α, hence so is the fundamental of the current, leading to a lagging power factor.

On the DC side, only the DC component of the voltage carries power, since there is no harmonic content in the current. On the AC side the power is carried only by the fundamental, since there are no harmonics in the voltage.

$$P = V_s I_{s1} \cos\alpha = V_d I_d \tag{8.11}$$

8.5.1 Inverter Mode

If somehow the current on the DC side is sustained even if the voltage reverses polarity, then power will be transferred from the DC to the AC side. The voltage on the D side can reverse polarity when

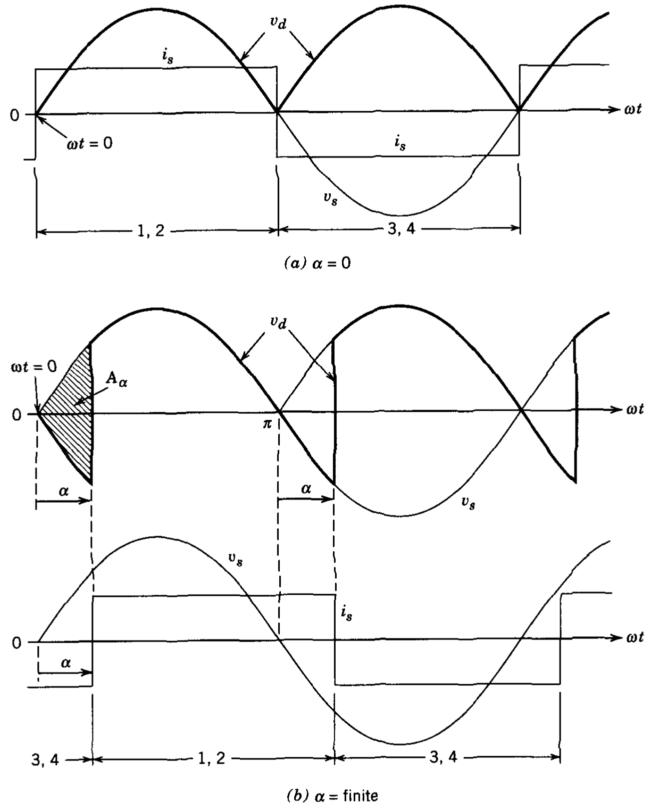


Fig. 8.9 Waveforms of One-phase full wave converter with Thyristors

the delay angle exceeds 90^0, as long as the current is maintained. This can only happen when the load voltage is as shown in figure 8.10, e.g. a battery.

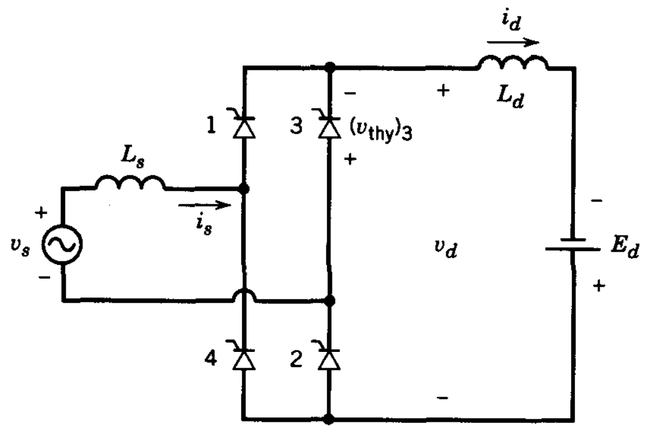


Fig. 8.10 Operation of a one-phase controlled Converter as an inverter

8.6 THREE-PHASE CONTROLLED CONVERTERS

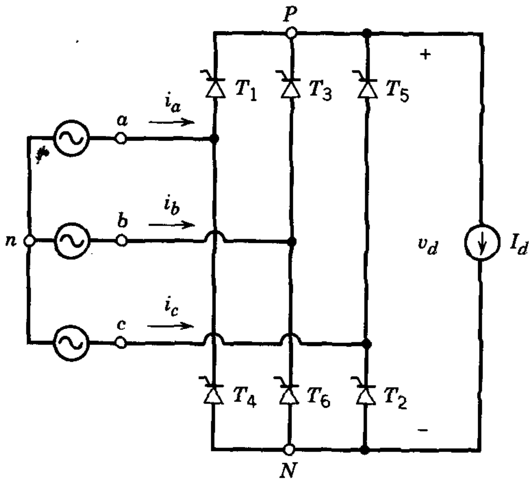


Fig. 8.11 Schematic of a three-phase Full-Wave Converter

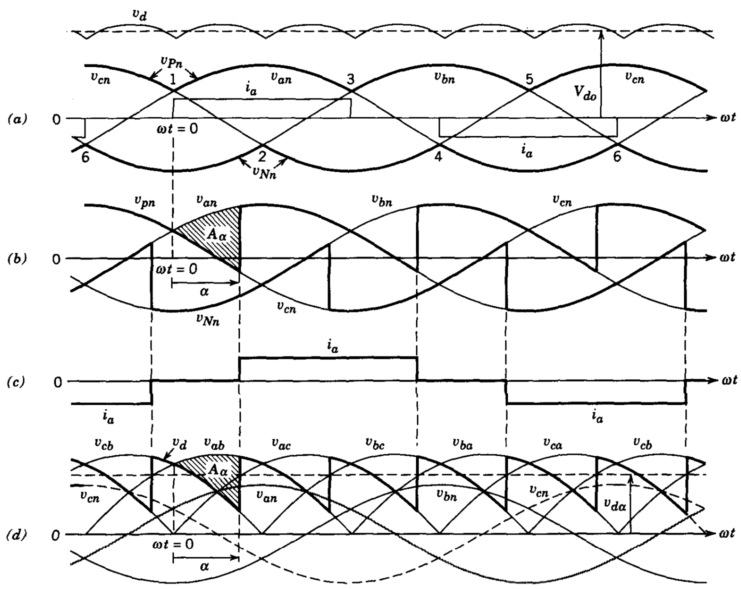


Fig. 8.12 Waveforms of a Three-phase Full-Wave Converter

As with diodes, only six thyristors are needed to accommodate three phases. Figure 8.11 shows the schematic of the system, and figure 8.12 shows the output voltage waveform. The delay angle α is again measured from the point that a thyristor becomes forward biased, but in this case the point is at the intersection of the voltage waveforms of two different phases. The voltage on the DC side is then:

$$V_{do} = \frac{3}{\pi}\sqrt{2}V_{LL}\cos\alpha = 1.35V_{LL}\cos\alpha \tag{8.12}$$

while the power for both the AC and the DC side is

$$P = V_d I_{do} = 1.35V_{ll} Id\cos\alpha = \sqrt{3}V_{ll}I_{s1}\cos\alpha \tag{8.13}$$

which leads to:

$$I_{s1} = 0.78I_d \tag{8.14}$$

Again if the delay angle α is extended beyond 90^0, the converter transfers power from the DC side to the AC side, becoming an inverter. We should keep in mind, though, that even in this case the converter is drawing reactive power from the AC side.

8.7 *NOTES

1. For both 1-phase and 3-phase controlled rectifier delay in α creates a phase displacement of the phase current with respect to the phase voltage, equal to α. The cosine of this angle is the power factor of the first harmonic.

2. For both motor and generator modes the controlled rectifier absorbs reactive power from the three-phase AC system, although it can either absorb or produce real power. It also needs the power line to commutate the thyristors. This means that inverter operation is possible only when the rectifier is connected to a power line.

3. When a DC motor or a battery is connected to the terminals of a controlled rectifier and α becomes greater then 90^0, the terminal DC voltage changes polarity, but the direction of the current stays the same. This means that in order for the rectifier to draw power from battery or a motor that operates as a generator turning in the same direction, the terminals haver to be switched.

9

Inverters

Although the AC-to-DC converters we have already studied can transfer power from the DC side to the AC system, they require the presence of such an AC system in order to commutate the thyristors and provide the required reactive power. In this chapter we'll study a similar system using devices that we can turn both on and off, like GTOs, BJTs IGBTs and MOSFETs, which allows the transfer of power from the DC source to any AC load. Figure 9.1 shows a typical application of a complete system, where the supply power of constant voltage and frequency is rectified, filtered and then inverted to provide an output of desired voltage and frequency.

We'll study first the operation of a single phase inverter and then we'll expand to three-phases.

9.1 1-PHASE INVERTER

Figure 9.2 shows the operation of on leg of the inverter regardless of the number of phases. To illustrate the point better, the input DC voltage is divided into two equal parts. When the upper switch T_{A+} is closed, the output voltage V_{Ao} will be $\frac{1}{2}V_d$, and when the lower switch T_{A-} is closed, it will be $-\frac{1}{2}V_d$. Deciding which switch to close in order to obtain a certain waveform will be determined by the PWM comparison shown in figure 9.3. We define as the frequency modulation index the ratio of the frequencies of the carrier (triangular wave) to the control signal:

$$m_f = \frac{f_s}{f_1} \tag{9.1}$$

and as amplitude modulation index:

$$m_a = \frac{V_{control}}{V_{tri}} \tag{9.2}$$

Two comments here:

1. The output voltage in figure 9.3 at first look does not resemble the expected waveform (i.e. the control signal). Its fundamental, though, does, and one can filter out the higher harmonics.

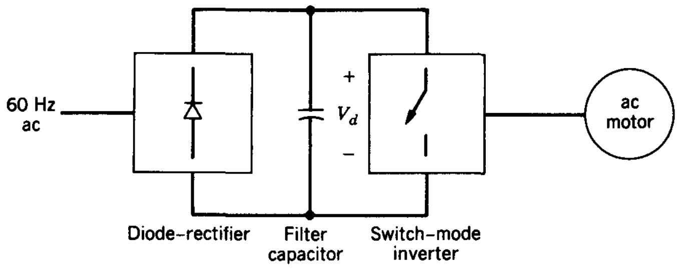


Fig. 9.1 Typical variable voltage and frequency system supplied from a power system

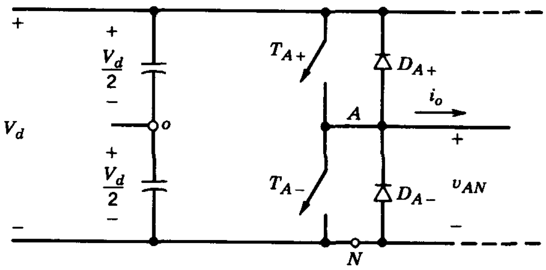


Fig. 9.2 One leg of an inverter

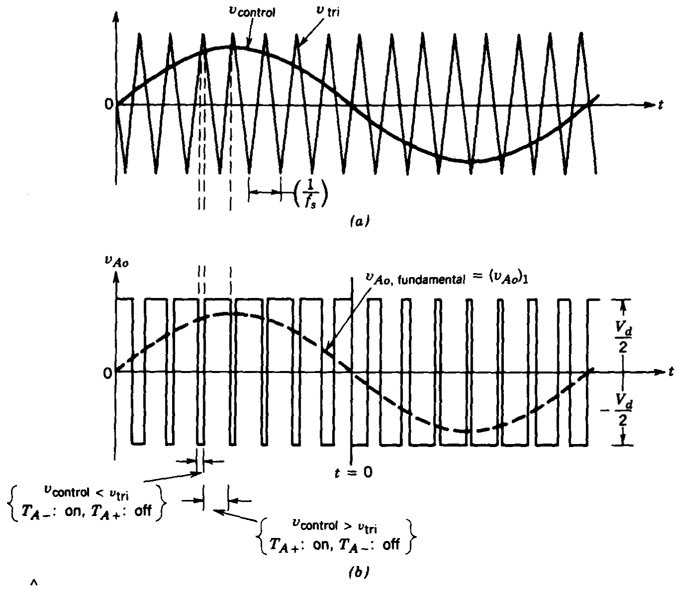


Fig. 9.3 PWM scheme to determine which switch should be closed

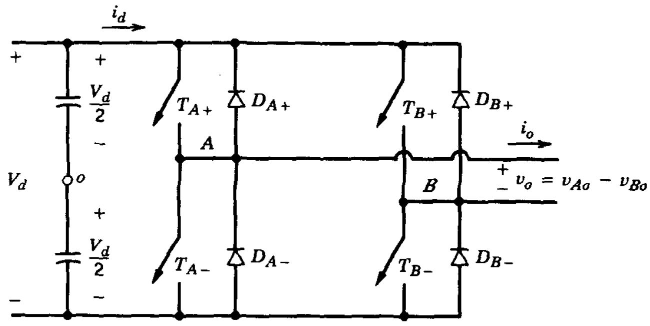


Fig. 9.4 One-phase full wave inverter

2. The switches in the inverter can conduct only in one direction. Inductive loads, though, require the current to continue to flow after a switch has been turned off. Allowing this current to flow is the job of the antiparallel diodes.

A full bridge inverter is shown in figure 9.4. It has four controlled switches, each with an antiparallel diode, and diagonally placed switches operate together. The output voltage will oscillate between $+V_d$ and $-V_d$ and the **amplitude** of the fundamental of the output voltage will be a linear function of the amplitude index $\hat{V}_o = mV_d$ as long as $m_a \leq 1$. Then the rms value of the output voltage will be:

$$V_{o1} = \frac{m_a}{\sqrt{2}} \frac{V_d}{2} = 0.353 m_a V_d \tag{9.3}$$

When m_a increases beyond 1, the output voltage increases also but not linearly with it, and can reach peak value of $\frac{4}{\pi} V_d$ when the reference signal becomes infinite and the output a square wave. Its RMS value, then will be:

$$V_{o1} = \frac{2\sqrt{2}}{\pi} \frac{V_d}{2} = 0.45 V d \tag{9.4}$$

Equating the power of the DC side with that of the AC side

$$P = V_d I_{d0} = V_{o1} I_{o1} pf \tag{9.5}$$

Hence for normal PWM

$$I_{d0} = 0.353 m_a I_{o1} pf \tag{9.6}$$

and for square wave

$$I_{d0} = 0.45 I_{o1} pf \tag{9.7}$$

9.2 THREE-PHASE INVERTERS

For three-phase loads, it makes more sense to use a three-phase inverter, rather than three one-phase inverters. Figure 9.5 shows a schematic of this system:

The basic PWM scheme for a three-phase inverter has one common carrier and three separate control waveforms. If the waveforms we want to achieve are sinusoidal and the frequency modulation index m_f is low, we use a synchronized carrier signal with m_f an integer and multiple of 3.

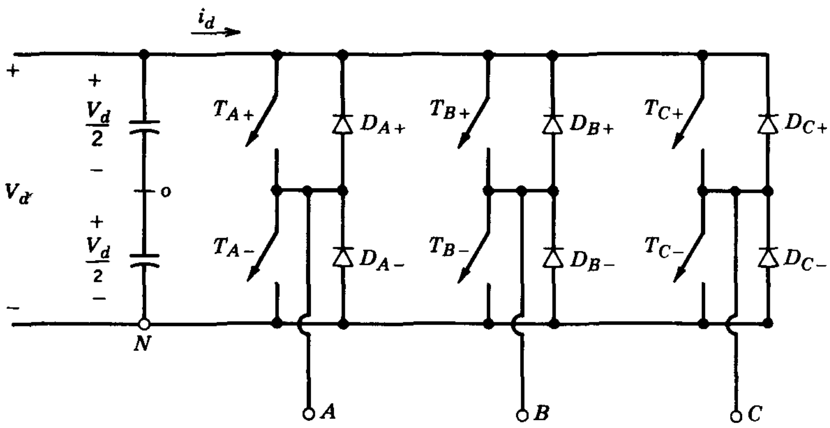


Fig. 9.5 Three-phase, full-wave inverter

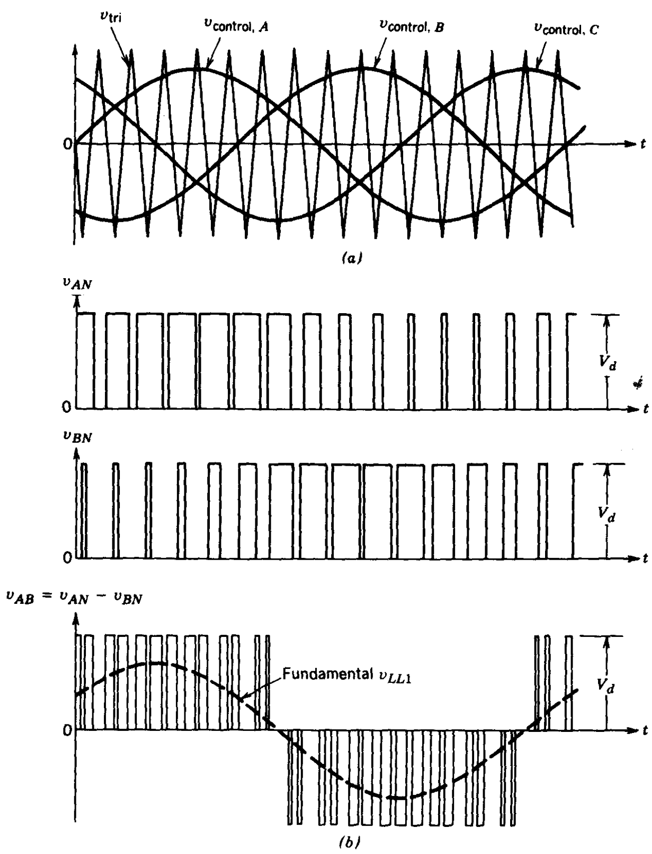


Fig. 9.6 Three-phase Pulse Width Modulation

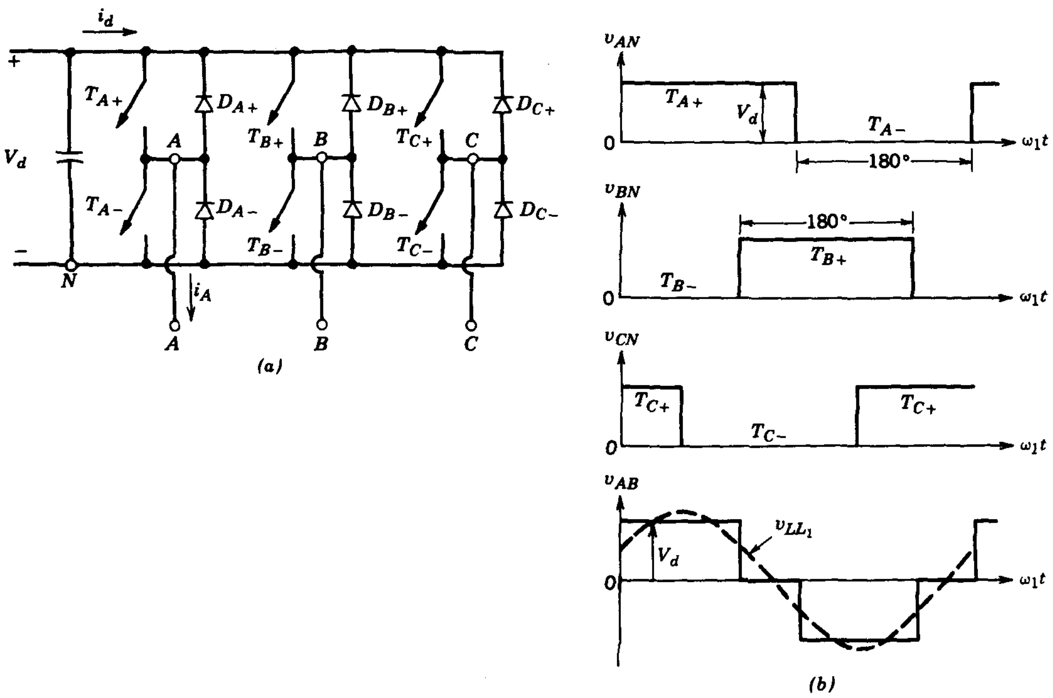


Fig. 9.7 6-step operation of a PWM inverter

As long as m_a is less than 1, the rms value of the fundamental of the output voltage is a linear function of it:

$$V_{LL1} = \frac{\sqrt{3}}{2\sqrt{2}} m_a V_d \simeq 0.612 m_a V_d \tag{9.8}$$

On the other hand, when the control voltage becomes infinite, the rms value of the fundamental of the output voltage becomes:

$$V_{LL1} = \frac{\sqrt{3}}{\sqrt{2}} \frac{4}{\pi} \frac{V_d}{2} \simeq 0.78 V_d \tag{9.9}$$

In this case the output voltage becomes rectangular and the operation is called 6-step operation, as shown in figure 9.7b.

Equating the power on the DC and AC sides we obtain: Equating the power of the DC side with that of the AC side

$$P = V_d I_{d0} = \sqrt{3} V_{ll1} I_{o1} pf \tag{9.10}$$

Hence for normal PWM

$$I_{d0} = 1.06 m_a I_{o1} pf \tag{9.11}$$

and for square wave

$$I_{d0} = 1.35 I_{o1} pf \tag{9.12}$$

Finally, there other ways to control the operation of an inverter. If it is not the output voltage waveform we want to control, but rather the current, we can either impose a fast controller on the voltage waveform, driven by the error in between the current signal and reference, or we can apply a hysteresis band controller, shown for one leg of the inverter in figure 9.8

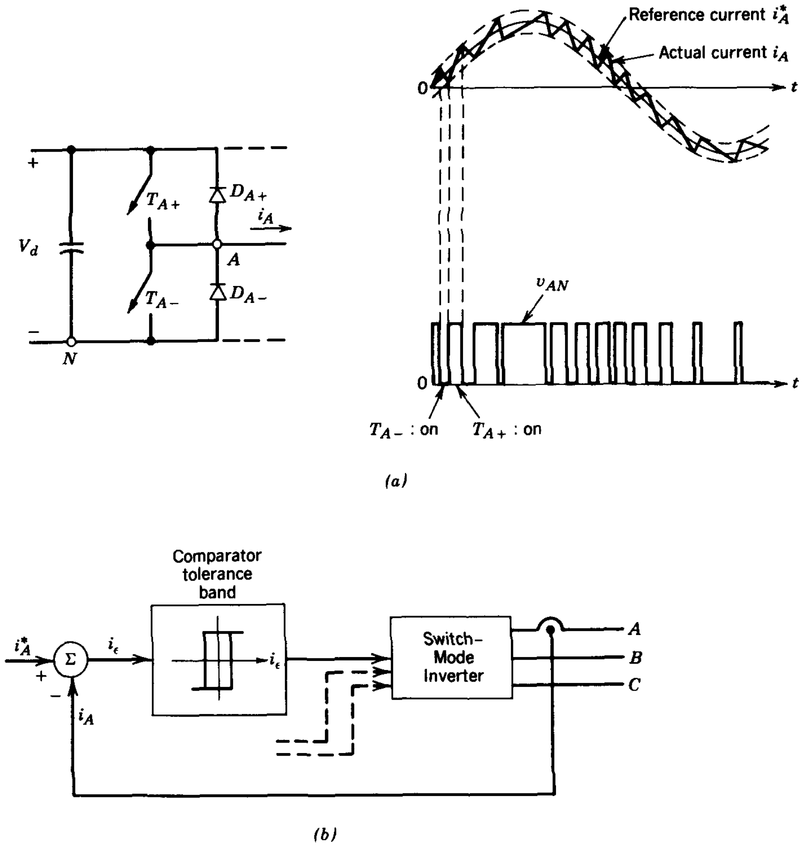


Fig. 9.8 Current control with hysteresis band

Notes

- With a sine-triangle PWM the harmonics of the output voltage s are of frequency around nf_s, where n is an integer and f_s is the frequency of the carrier (triangle) waveform. The higher this frequency is the easier to filter out these harmonics. On the other hand, increasing the switching frequency increases proportionally the switching losses. For 6-step operation of a 3-phase inverter the harmonics are even except the triplen ones, i.e. they are of order 5, 7, 11, 13, 17 etc.
- When the load of an inverter is inductive the current in each phase remains positive after the voltage in that phase became negative, i.e. after the top switch has been turned off. The current then flows through the antiparallel diode of the bottom switch, returning power to the DC link. The same happens of course when the bottom switch is turned off and the current flows through the antiparallel diode of the top switch.

9.2.1 Example

A 3-phase controlled rectifier is supplying a DC motor with $k = 1Vs$ and $R = 1\Omega$. The rectifier is fed from a $208V\ l - l$ source.

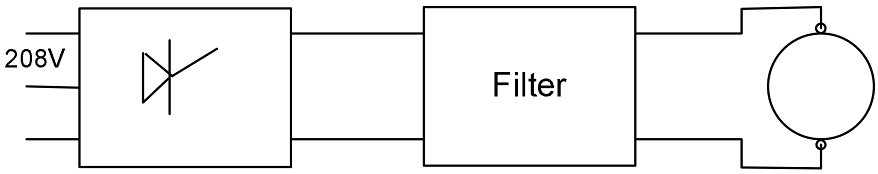


Fig. 9.9 figure for 9.2.1

1a *Calculate the maximum no-load speed of the DC motor.*
Without load the current is zero. Hence:

$$V = k\omega + IR = k\omega$$

The maximum speed is then determined by the maximum DC voltage:

$$V_{max} = k\omega_{max}$$

This maximum DC voltage is provided by the controlled rectifier for $\alpha = 0$:

$$V_{max} = 1.35V_{ll} = 281.8V$$

hence

$$\omega_{max} = 280.8rad/s$$

1b *The motor now is producing torque of $20Nm$. What is the maximum seed the motor can achieve?*
Now that there is load torque there is current:

$$T = kI \Rightarrow I = 20Nm$$

Again

$$\omega = \frac{V - IR}{k} = \frac{280.8 - 20 \cdot 1}{1} = 260.8rad/s$$

1c *For the case in 1b calculate the total rms current the first harmonic and the power factor at the AC side.*

The fundamental of the AC current is

$$I_{s1} = 0.78 I_d = 15.6A$$

Power factor is then 1.

1d *The motor is now connected as* a generator, *with a counter torque of* $20Nm$ *at* $1500rpm$. *What should be the delay angle and AC current?*

For a DC generator

$$V = k\omega - IR = k\omega - \frac{T}{k}R1 \cdot 1500\frac{2\pi}{60} - \frac{2}{1}1 = 137.08V$$

Since this is a generator this voltage is negative for the inverter (see notes)

$$-137,08 = 1.35 \cdot 208\cos\alpha \Rightarrow \cos\alpha = -0.488 \Rightarrow \alpha = 119.22^0$$

9.2.2 Example

In the system below the AC source is constant. The load voltage is $150V(l-l)$, $20Am$ $52Hz$, $0.85pf\ lagging$. *Calculate:*

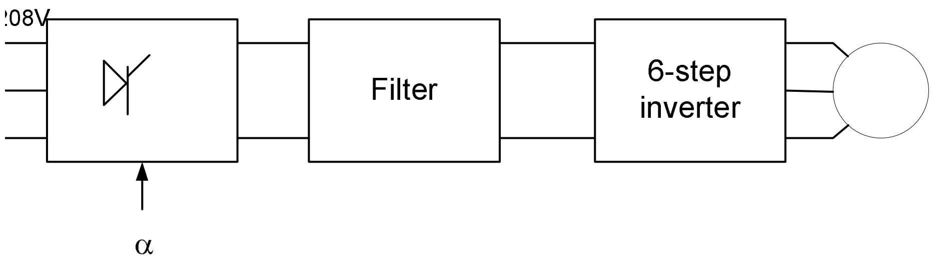


Fig. 9.10 figure for 9.2.2

a *The voltage on the DC side and the DC component of the current.*

For 6-step inverter

$$V_{ll,1} = 0.78V_d \Rightarrow V_d = 192V$$

$$P = \sqrt{3}V_{ll}I_l pf = V_d I_{d0} \Rightarrow I_{d0} = \frac{1.35 \cdot 150 \cdot 20 \cdot 0.85}{192} = 23A$$

b *Calculate the source side (208V AC) rms and fundamental current and power factor.*

For a 3-phase rectifier

$$V_d = 1.35V_{ll}\cos\alpha \Rightarrow 192 = 1.35 \cdot 208 \cdot \cos\alpha \Rightarrow \cos\alpha = 0.685$$

$$I_{s1} = 0.78I_d = 17.94A$$

$$pf = \cos\alpha = 0.685\ lagging$$

10

DC-DC Conversion

We will study DC to DC converters operating under certain conditions. The use of such converters are extensive in automotive applications, but also in cases where a DC voltage produced by rectification is used to supply secondary loads. The conversion is often associated with stabilizing, i.e. the input voltage is variable but the desired output voltage stays the same. The converse is also required, to produce a variable DC from a fixed or variable source. The issues of selecting component parameters and calculating the performance of the system will be addressed here. Since these converters are switched mode systems, they are often referred to as choppers.

10.1 STEP-DOWN OR BUCK CONVERTERS

The basic circuit of this converter is shown in figure 10.1 connected first to a purely resistive load. If we remove the low pass filter shown and the diode the output voltage $v_o(t)$ is equal to the input voltage V_d when the switch is closed and to zero when the switch is open, giving an average output voltage V_o:

$$V_o = \frac{1}{T_S}\left[\int_0^{t_{on}} V_d dt + \int_{t_{on}}^{T_s} 0 dt\right] = \frac{t_{on}}{T_s} V_d \tag{10.1}$$

with $t_{on}/T_s = D$, the duty ratio.

The low pass filter attenuates the high frequencies (multiples of the switching frequency) and leaves almost only the DC component. The energy stored in the filter inductor (or the load inductor) has to be absorbed somewhere other than the switch, hence the diode, which conducts when the switch is open.

We'll study this converter in the continuous mode of operation i.e. the current through the inductor never becomes zero. As the switch opens and closes the circuit assumes one of the topologies of figure 10.2.

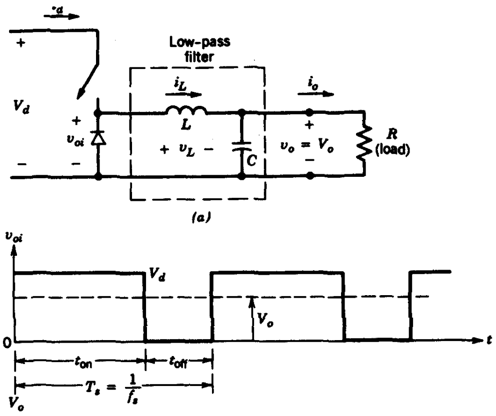


Fig. 10.1 Topology of the buck chopper

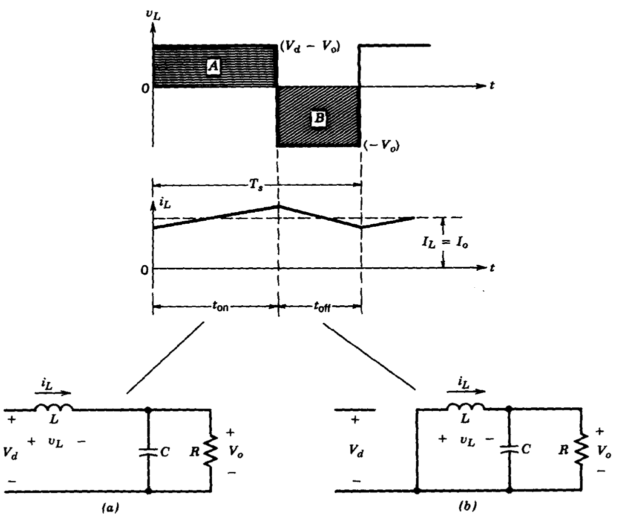


Fig. 10.2 Operation of the buck chopper

We'll use the fact that the average voltage across the inductor is zero. Assuming perfect filter, the voltage across the inductor is V_d during t_{on} and $-V_o$ the remaining of the cycle. Hence:

$$\int_0^{t_{on}} (V_d - V_o)dt + \int_{t_{on}}^{T_s} (-V_o)dt = 0 \tag{10.2}$$

$$\Rightarrow (V_d - V_o)t_{on} - V_o(T_s - t_{on}) = 0 \tag{10.3}$$

$$\Rightarrow \frac{V_o}{V_d} = \frac{t_{on}}{T_s} = D \tag{10.4}$$

A second consideration is that the input and output powers are the same, hence:

$$V_d I_d = V_o I_o \tag{10.5}$$

$$\Rightarrow \frac{I_o}{I_d} = \frac{V_d}{V_o} = \frac{1}{D} \tag{10.6}$$

Note that in discontinuous mode the output DC voltage is less that that given here, and the chopper less easy to control.

At the boundary between continuous and discontinuous mode, the inductor current reaches zero for one instant every cycle, as shown in figure 10.3a. Using this figure we can see that at this operating point, the average inductor current is $I_L = \frac{1}{2}\hat{i}_L$. Further studying the geometry we obtain:

$$I_L = \frac{1}{2} t_{on}(V_d - V_o) = \frac{DT_S}{2L}(V_d - V_o) \tag{10.7}$$

Since the average inductor current is the average output current (the average capacitor current is obviously zero), equation 10.3 defines the minimum load current current that will sustain continuous conduction.

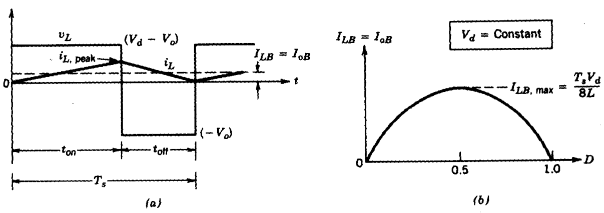


Fig. 10.3 Operation of the buck Converter at the boundary of Continuous Conduction

Finally a consideration is the output voltage ripple. We assume that the ripple current is absorbed by the capacitor, i.e. the voltage ripple is small. The ripple voltage is then due to the deviation from the average of the inductor current as shown in figure 10.4. Under these conditions:

$$\Delta V_0 = \frac{\Delta Q}{C} = \frac{1}{L}\frac{1}{2}\frac{\Delta I_L}{2}\frac{T_s}{2} \tag{10.8}$$

$$\text{where} \qquad \Delta I_L = \frac{V_o}{L}(1 - D)T_s \tag{10.9}$$

$$\Rightarrow \frac{\Delta V_o}{V_o} = \frac{1}{8}\frac{T_s^2}{LC}(1 - D) \tag{10.10}$$

Another way to look at this is to define the switching frequency $f_s = 1/T_s$ and use the corner frequency of the filter, $f_c = 1/(2\pi\sqrt{LC})$:

$$\frac{\Delta V_o}{V_o} = \frac{\pi^2}{2}(1 - D)\left(\frac{f_c}{f_s}\right)^2 \tag{10.11}$$

10.2 STEP-UP OR BOOST CONVERTER

Here the output voltage is always higher than the input. The topology is shown in figure 10.5.

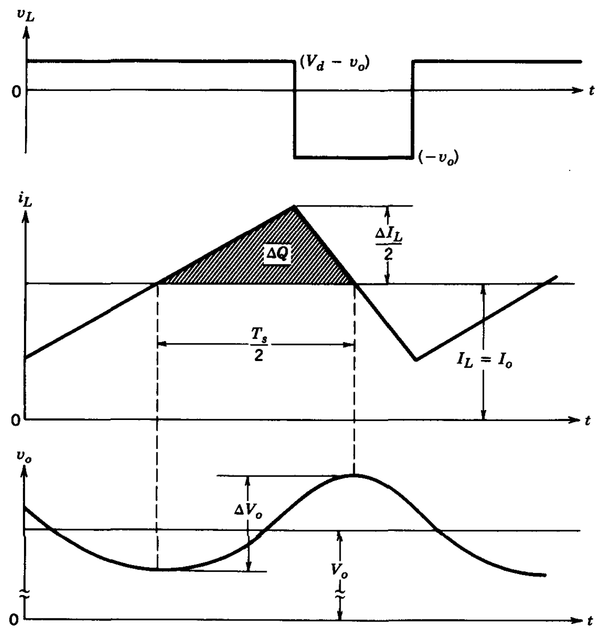


Fig. 10.4 Analysis of the output voltage ripple of the buck Converter

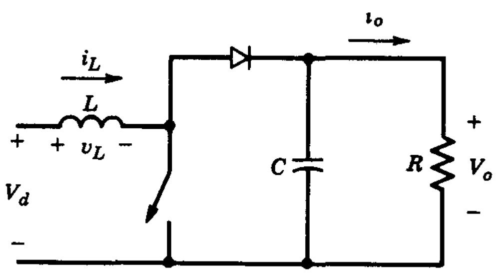


Fig. 10.5 Schematic Diagram of a Boost Converter

There are two different topologies, based on the condition of the switch, as shown in figure 10.6 Again, the way to calculate the relationship between input and output voltage we have to take the average current of the inductor to be zero, and the output power equal to the input power hence:

$$V_d t_{on} + (V_d - V_o)(T_s - t_{on}) = 0 \tag{10.12}$$

$$\Rightarrow \frac{V_o}{V_d} = \frac{1}{1-D} \tag{10.13}$$

$$\Rightarrow \frac{I_o}{I_d} = 1 - D \tag{10.14}$$

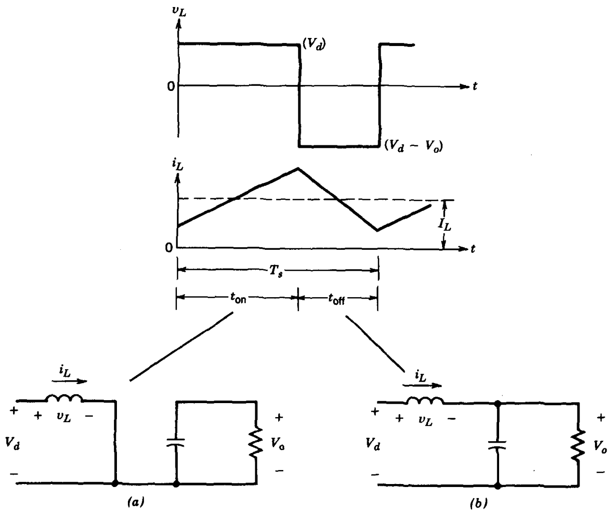


Fig. 10.6 Two Circuit Topologies of the boost Converter

To determine the values of inductance and capacitance we will study the boundary of continuous conduction like before and the output voltage ripple.

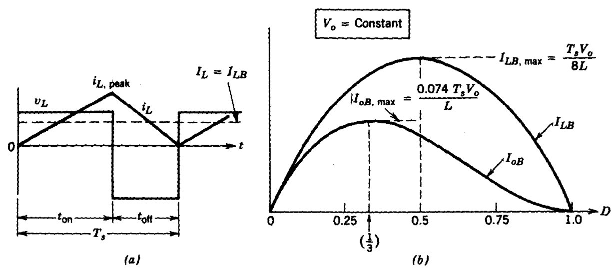


Fig. 10.7 The boundary between Continuous and Discontinuous Conduction of a Boost Converter

At the boundary of the continuous conduction, as shown in figure 10.7, the geometry of the current waveform will give:

$$I_o = \frac{T_s V_o}{2L} D(1-D)^2 \tag{10.15}$$

The output current has to exceed this value for continuous conduction. Looking at the geometry of figure 10.8 and following an analysis similar to that of a buck converter we find that:

$$\frac{\Delta V_o}{V_o} = \frac{DT_s}{RC} \tag{10.16}$$

It is important to note that the operation of a boost converter depends on parasitic components, especially for duty cycle approaching unity. These components will limit the output voltage to levels well below those given by the formula 10.13.

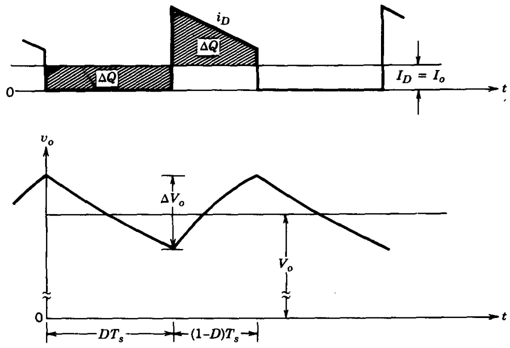


Fig. 10.8 Calculating the output voltage ripple for a boost inverter

10.3 BUCK-BOOST CONVERTER

This converter, the topology of which is shown in figure 10.9, can provide output voltage that can be lower or higher than that of the input.

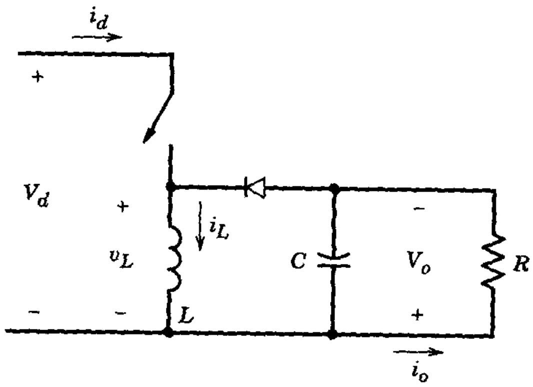


Fig. 10.9 Basic buck-boost converter

Again the operation of the converter can be analyzed using the two topologies resulting from operation of the switch, shown in figure 10.10.

By equating the integral of the inductor voltage to zero we can get:

$$V_d D T_s + (-V_o)(1 - D)T_s = 0 \tag{10.17}$$

$$\Rightarrow \frac{V_o}{V_d} = \frac{D}{1 - D} \tag{10.18}$$

At the boundary between continuous and discontinuous conduction we can use figure 10.11 to find that

$$I_o = \frac{T_s V_o}{2L}(1 - D)^2 \tag{10.19}$$

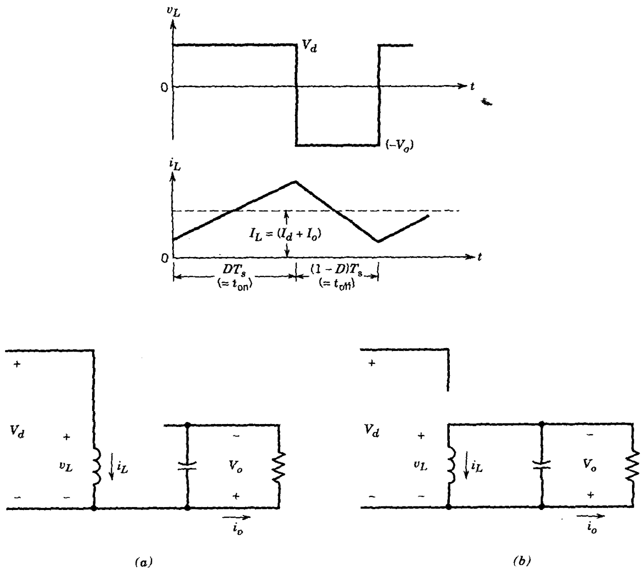


Fig. 10.10 Operation of a buck boost chopper

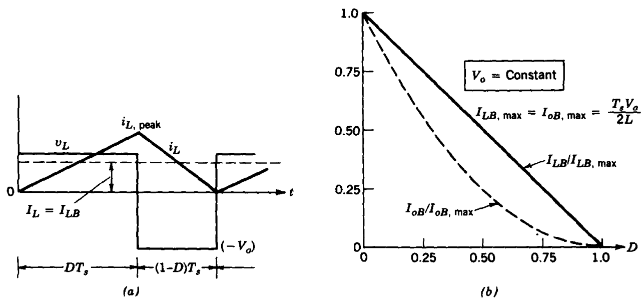


Fig. 10.11 Operation of a buck boost chopper

The output voltage ripple, as calculated based on figure 10.12 is

$$\frac{\Delta V_o}{V_o} = D\frac{T_s}{RC} \tag{10.20}$$

10.3.1 Example

The input of a step down converter varies from $30V$ to $40V$ and the output voltage is to be constant $20V$, with output power varying between $100W$ and $200W$. The switch is operating at $20kHz$. What is the inductor needed to keep the inductor current continuous? What is then the filter capacitor needed to keep the output ripple below 2%.

The duty cycle will vary between $D_1 = 20/30 = 0.667$ and $D_2 = 20/40 = 0.5$. The load current will range between $I_{o1} = 100/20 = 5A$ and $I_{o2} = 200/20 = 10A$.

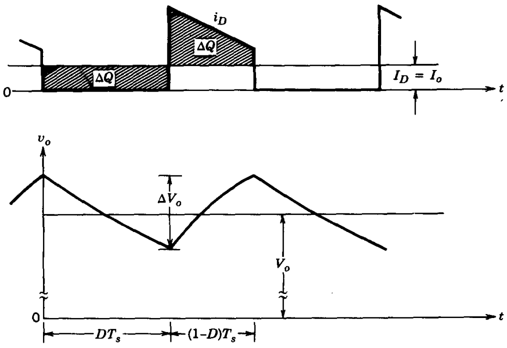


Fig. 10.12 Calculating the output voltage ripple for a boost inverter

The minimum current needed to keep the inductor current continuous is

$$I_{o\ min} = \frac{DT_s}{2L}(V_d - V_o)$$

since the constant is the output voltage V_o and the minimum *load current has to be greater than $I_{o\ min}$, we'll express it as a function of V_o and make it less or equal to $5A$!*

$$5A \geq I_{o\ min} = \frac{DT_s}{2L}(V_d - V_o) = \frac{V_oT_s}{2L}(1 - D)$$

$T_s = 1/20kHz$, $V_o = 20V$ *and the max value is achieved for* $D = 0.5$, *leading to* $L_{min} = 50\mu H$. *As about the ripple, the highest will occur at* $1 - D = 0.5$. *Hence:*

$$0.02 = \frac{\pi^2}{2}0.5\left(\frac{f_c}{10\cdot 10^3}\right)^2 \Rightarrow f_c = 900Hz$$

$$\Rightarrow \frac{1}{2\pi\sqrt{50\cdot 10^{-}6C}} = 900$$

$$\Rightarrow C = 625\mu F$$